nance model, for they undoubtedly have something to do with the Pomeranchuk singularity. Although the latter is specifically left out of the dual-resonance model, its effect may be partly included, for unitarity is used in arriving at (6). We have, however, no interpretation to offer at present.

Our findings bear on some recent speculations by Chen and Harte[13] and by Berman and Jacob.[14] They extended the Wu-Yang conjecture[15]

$$\sigma_{ep\to ep}\approx(\sigma_{\rm el})^{1/2} \tag{14}$$

to the inclusive reaction $pp\to p+$anything, and suggested that in part of the quasielastic region, namely, that for which M^2/t remains fixed as $r\to 1$,

$$d^2\sigma/dqd\Omega\approx(\sigma_{\rm el})^{1/2}R, \tag{15}$$

where R is a slowly varying function related to the cross section for $e+p\to e+$anything. This conjecture is not borne out by our result, which resembles more closely (15) with $(\sigma_{\rm el})^{1/2}$ replaced by $\sigma_{\rm el}$.

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Flat Proton Spectrum and Its Implications*

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The longitudinal momentum distribution of the final detected proton in pp inclusive reactions is flat for energies ranging from 20 to 1500 GeV. We show how this can be understood in the diffractive model, and how this information provides a way for calculating the average multiplicity and the pion longitudinal momentum distributions in pp inclusive reactions. No arbitrary parameter is used. The agreement with experiment is remarkably good.

Proton spectrum.—It has been known for some time that up to 30 GeV/c, the c.m. longitudinal momentum distribution $d\sigma/dp_{\parallel}^*$ of the proton in the inclusive reaction $p+p\to p+$anything is independent of $p_{\parallel}^*$.[1-3] It is dramatically different from the pion spectra and has always been somewhat of a mystery. Recent results[4] from the CERN intersecting-storage-rings experiment show that this property persists up to 1500 GeV/c; moreover, the height of the proton spectrum is consistent with the invariant distribution being "limiting."[5] We show below how these characteristics can be understood within the framework of the diffractive model in which the multiparticle production amplitude can be described by the exchange of a single Pomeranchukon between two clusters.[6,7]

The mean transverse momentum $\langle k_\perp^2\rangle^{1/2}$ of the secondary pions in hadron-hadron collisions is known to be not too sensitive to the nature of the hadrons, to the number of prongs involved, to the beam momentum, and to their own longitudinal momenta. In the diffractive-excitation picture in which each of the excited hadrons is assumed to decay isotropically in its respective rest frame, it is therefore suggestive that pions are emitted with a definite average energy E that depends on $\langle k_\perp^2\rangle^{1/2}$ but is independent of the number of pions emitted, etc. Thus, *for each collision* the final state can be thought of as forming two clusters of pions although the collection of many events does not have this form. For simplicity we shall ignore here the production of strange particles and antibaryons since their production cross sections are low. In the decay of the excited state the proton should, on the average, form a "nucleus" for the pionic radiation; and hence, its rest frame should coincide with the cluster frame.

With these assumptions we can obtain the c.m. longitudinal momentum $p_\parallel^*$ of the proton by Lorentz transformation. For given masses of the two clusters, M_1 and M_2, the c.m. momentum of the clusters, q^*, is fixed; from it the velocities of the clusters (hence, of the protons) can be determined. Thus, for the first cluster

$$q^* = \gamma v M_1, \quad \gamma = (1-v^2)^{-1/2}, \tag{1}$$

$$p_\parallel^* = \gamma v M, \tag{2}$$

where M is the mass of the proton. Let P be the incident c.m. momentum, and define the scaled variable $y_1 = p_\parallel^*/P$. We are primarily interested in the kinematical region where $M_i \ll P$ [as will become evident in (7) below]; thus, we may approximate $q^* \approx P$. We then obtain from (1) and (2)

$$y_1 = M/M_1 \quad (\text{similarly } y_2 = M/M_2). \tag{3}$$

Now, let n_1 denote the number of pions in the (first) cluster. The proton being at rest in the cluster frame implies

$$M_1 = M + n_1 E, \tag{4}$$

which in conjunction with (3) yields

$$y_1^{-1} - 1 = n_1/\xi, \quad \xi \equiv M/E. \tag{5}$$

The flatness of the proton spectrum[8] implies

$$d\sigma/dy_1 = A, \tag{6}$$

where A is independent of y_1 and only weakly dependent on the incident energy. In terms of M_1, (6) can be expressed as

$$d\sigma/dM_1 = -AM/M_1^2. \tag{7}$$

We now see how (7) can be derived in the diffractive model. The two clusters of particles are produced, in Regge language, with the exchange of the Pomeranchukon which in our model is a fixed singularity at $j=1$.[9,10] Assuming that the Pomeranchukon is factorizable, we need further a knowledge of the behavior of the Pomeranchukon-hadron amplitude at large values of the cluster masses. Here we either assume on phenomenological ground that the triple-Pomeranchukon coupling is weak[11] (consequently the f trajectory dominates) or assert as a theoretical conjecture that the notion of duality applies also to the Pomeranchukon-hadron "scattering" amplitude. In the latter case the interpretation of the excited state before its decay as a Regge recurrence of the incident hadron implies that it is dual to the leading non-Pomeranchukon Regge pole, i.e., the f trajectory. Furthermore, duality means that the imaginary part of the forward Pomeranchukon-hadron scattering amplitude behaves as $M_i^{2\alpha_f(0)}$, $i=1,2$, even for nonasymptotic values of M_i, such as the typical resonance masses. With s denoting the square of the c.m. energy and $-t$ the square of the momentum transfer carried by the Pomeranchukon, then in the limit $s \gg (M_1M_2/M)^2$ we have

$$\frac{d^3\sigma}{dM_1^2 dM_2^2 dt} = s^{-2}\beta^2(t)\left(\frac{s}{M_1^2M_2^2}\right)^{2\alpha_P(t)}(M_1^2M_2^2)^{\alpha_f(0)}. \tag{8}$$

Here $\beta(t)$ has the exponential peak characteristic of diffraction. Setting $\alpha_P(t)=1$ and $\alpha_f(0)=\frac{1}{2}$ in (8) we obtain

$$d\sigma/dM_1^2 \propto M_1^{-3}, \tag{9}$$

which agrees with (7). This explains the flatness of the proton spectrum. The validity of (8) for nonasymptotic values of M_i as suggested by duality implies that the spectrum is flat not only for small values of y_i but also for most of the y_i range (roughly up to M/M^*, where M^* is a typical resonance mass). To understand the empirical fact that the flatness extends almost all the way to $y_i=1$ would perhaps require a more refined version of our model. It is evident, however, that the simple picture described here has captured the essence of the mechanism.

Average multiplicity.—The average pion multiplicity in pp collisions is

$$\langle n\rangle = \int \frac{d^2\sigma}{dn_1 dn_2}(n_1+n_2)\,dn_1\,dn_2 \Big/ \int \frac{d^2\sigma}{dn_1 dn_2}\,dn_1\,dn_2. \tag{10}$$

The boundaries of integration will be taken to be $n_1+n_2=1$ (at least one pion must be produced in an inelastic event) and $n_1+n_2=(\sqrt{s}-2M)/E$, the latter being the maximum number of pions that can be produced with the available energy. Using (5) and (6), the integrals in (10) may be evaluated. Assuming that the probability of finding charged pions is $\frac{2}{3}$ when the total pion charge is zero, and that protons and neutrons occur equally frequently in the final states, the total charge multiplicity, including those of the proton tracks is[12] $\langle n_c\rangle = \frac{1}{3}(2\langle n\rangle + 4)$. This leads to

$$\begin{aligned}
&\langle n_c\rangle = \tfrac{4}{3}(1-\xi+\xi N/D),\\
&N = (1-2M/\sqrt{s})\ln(\sqrt{s}/M-1) - (1+2\xi)^{-1}\ln(1+\xi^{-1}),\\
&D = 2[-M/\sqrt{s} - (M^2/s)\ln(\sqrt{s}/M-1) + (\xi/2\xi+1) + (\xi/2\xi+1)^2\ln(1+\xi^{-1})].
\end{aligned} \tag{11}$$

Asymptotically this grows as $\ln s$. Note that both the slope and the height are completely determined by the proton mass M and the average pion energy E in the cluster frame. There are no free parameters. To get $E=\langle k_0\rangle$, we average $(k^2+\mu^2)^{1/2}$ over an isotropic Gaussian momentum distribution, $\exp(-k^2/\langle k_\perp^2\rangle)$. Here and in the following we take[13] $\langle k_\perp^2\rangle^{1/2} = 350$ MeV/c; this yields[14] $\xi = M/E = 2.18$. The prediction according to (11) is plotted in Fig. 1 and compared with experimental data.[15] The agreement is good considering that no free parameter has been used.

Pion spectrum in pp collisions.—Let $x = k_\parallel^*/P$ be the pion c.m. longitudinal momentum scaled by P. The differential cross section $d\sigma/dx$ is

$$\frac{d\sigma}{dx} = \int \frac{d\sigma}{dn_1} n_1 g(n_1;x)\,dn_1, \tag{12}$$

where $g(n_1;x)$ is the probability density (whose integral over x is unity) of finding a pion with longitudinal momentum xP in a cluster of n_1 pions.

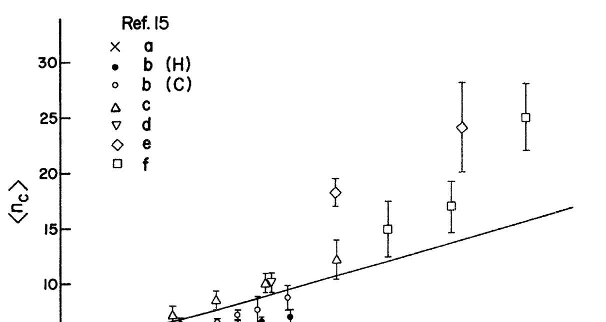

FIG. 1. Average charge multiplicity as a function of incident energy. Solid curve is the theoretical prediction.

Since the invariant pion spectrum at existing machine energies is limiting already,[4] we shall calculate (12) at infinite energy. The upper limit of the integral is then ∞; the lower limit n_0 is 1 or 2 according to whether the spectrum is that of π^+ or π^-. The probability function $g(n_1;x)$ describing an isotropic decay is assumed to be the Gaussian distribution $\exp(-k^2/\langle k_\perp^2\rangle)$ in the cluster frame. Since the pion c.m. momentum $k_\parallel^*$ and its momentum $k_\parallel$ in the cluster frame are related by a Lorentz transformation[16]

$$k_\parallel^* = \gamma(k_\parallel + E), \tag{13}$$

Eq. (2) implies that (after integration over $k_\perp$ and with stated normalization)

$$g(n_1;x) = \left(\frac{\alpha}{\pi}\right)^{1/2}\frac{\xi}{y}\exp\left[-\alpha\left(\frac{\xi x}{y}-1\right)^2\right], \qquad \alpha \equiv E^2/\langle k_\perp^2\rangle. \tag{14}$$

Substituting (5), (6), and (14) into (12), we obtain at infinite energy

$$F(x) = x\frac{d\sigma}{dx} = \xi A\left(\frac{\alpha}{\pi}\right)^{1/2}(G_1 - \xi x G_2), \tag{15}$$

$$G_1 = \frac{1}{2}\left(\frac{\pi}{\alpha}\right)^{1/2}\{1-\mathrm{erf}(\sqrt{\alpha}[(\xi+n_0)x-1])\},$$

$$G_2 = \int_{(\xi+n_0)x}^{\infty}\exp[-\alpha(z-1)^2]\,dz/z.$$

Again we note that apart from normalization $F(x)$ is completely determined by M, E, and n_0 without any free parameters. We have computed $F(x)$ for π^+ ($n_0=1$) and π^- ($n_0=2$) spectra which are shown by solid lines in Fig. 2. The curves are normalized to unity at $x=0.2$, and the data[2,4,17] with similar normalization are shown for comparison of the x dependences. Where our model is reliable, i.e., $x<0.5$, the agreement is remarkably good.

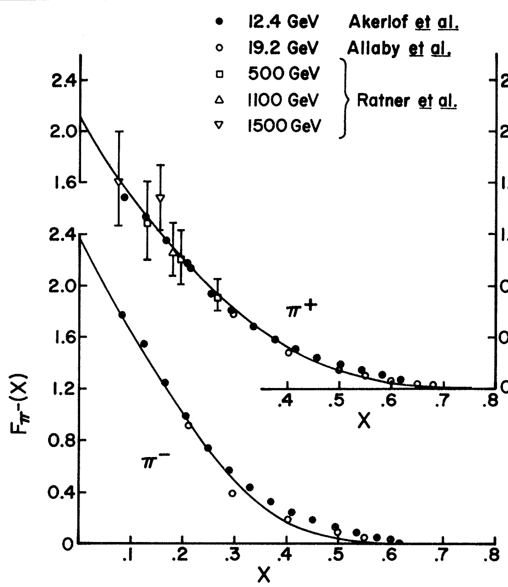

FIG. 2. Pion spectra in $p+p\rightarrow\pi^{\pm}+$anything. Solid curves are theoretical predictions. Data are taken from Refs. 2, 4, and 17. Everything is normalized to unity at $x=0.2$. The data points are all for $k_\perp^2$ in the range from 0.16 to 0.22 $(\mathrm{GeV}/c)^2$. Their dependences on $k_\parallel^*$ are essentially the same as the results after integrations over $k_\perp$.

For $x>0.5$ the cluster mass is low, so our statistical treatment of the decay is expected to be inadequate. In the wee-x region, $|x|\lesssim(4\langle k_\perp^2\rangle/s)^{1/2}$, although our $F(0)$ is finite, its magnitude, as well as its shape, is not reliable because in that region the two clusters are expected to have a considerable overlap in momentum space and should not be treated as completely independent of each other.

The pion spectra in Kp and πp collisions can similarly be calculated. Because of the lack of space here we describe them in a future publication, mentioning here only that the asymmetry in the x distribution can be readily understood within this model.

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