[22]Further details will be given elsewhere but it may be remarked here that the resulting distribution of ft values reproduces very well that found in practice across the two shells.

[23]$A = 24$ has $\log ft = 6.1$. The Monte Carlo computation showed that, as expected, large ft values tend to entrain large (positive or negative) values of δ/δ_0 and this effect may be operative here.

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Structure of the Electromagnetic Field in a Spatially Dispersive Medium*

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The exact mode expansion is derived for the electromagnetic field in a spatially dispersive model dielectric occupying the volume $0 \leq z \leq d$. The dispersion relations for the transverse as well as the longitudinal waves are deduced and the nature of the modes is briefly discussed.

Electrodynamics of spatially dispersive media, i.e., of media whose response to an incident electromagnetic field is spatially nonlocal, has attracted a great deal of attention since Pekar[1] predicted some rather remarkable phenomena associated with spatial dispersion. The close connection between this subject and the theory of excitons is, of course, well known.[1-4]

In spite of the great deal of interest in this subject, some rather basic questions in this domain have as yet not been solved. One of them concerns the exact mode expansion of an electromagnetic field in a spatially dispersive medium that does not occupy the whole infinite space. It is often assumed that in any volume occupied by a spatially dispersive medium the electromagnetic field may be expanded in terms of plane waves whose (generally complex) propagation vectors are identical with those appropriate to a field in a spatially dispersive medium occupying all space. That this assumption is questionable is clear if one recalls that in a half-space, even in the absence of any material medium, plane waves may be propagated that cannot be physically realized in the whole empty space. These are the so-called evanescent waves, well known in the theory of total internal reflection[5] and in connection with other interaction problems.[6]

In the present paper we derive an exact mode expansion for the field in a spatially dispersive model dielectric that occupies the volume $-\infty \leq x \leq \infty$, $-\infty \leq y \leq \infty$, $0 \leq z \leq d$. The exact dispersion relations for both transverse and longitudinal modes are found, and the nature of the expansion is briefly discussed. This mode expansion has a bearing on many aspects of the electrodynamics of spatially dispersive media and on the theory of excitons. We will show in another publication that our expansion leads readily to the exact solution of the problem of refraction and reflection on a half-space filled with a spatially dispersive medium and that the solution provides complete resolution of a long-standing controversy about the so-called additional boundary conditions,[4,7-9] generally believed to be necessary for the solution of this problem.

Consider first an electromagnetic field in a spatially dispersive medium occupying the whole infinite space. For the sake of simplicity we assume the medium to be homogeneous and nonmagnetic. The constitutive relation which couples the electric vector $\vec{E}$ and the electric displacement $\vec{D}$ may be expressed in the form

$$\hat{\vec{D}}(\vec{k}, \omega) = \hat{\epsilon}(\vec{k}, \omega)\hat{\vec{E}}(\vec{k}, \omega), \tag{1}$$

where the circumflex denotes a four-dimensional Fourier transform [with kernel $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$]. Following Hopfield and Thomas,[4] we restrict our discussion, for the sake of simplicity, to a medium for which the dielectric constant $\hat{\epsilon}(\vec{k}, \omega)$ is of the form

$$\hat{\epsilon}(\vec{k}, \omega) = \epsilon_0(\omega) + \alpha_e \omega_e^2 [\omega_e^2 - \omega^2 + (\hbar\omega_e/m_e^*)k^2 - i\omega\Gamma_e]^{-1}. \tag{2}$$

Here, $\epsilon_0(\omega)$ is the wave-vector–independent background dielectric constant associated with all transitions other than the exciton transition at frequency ω_e, α_e is the oscillator strength associated with the exciton transition, $m_e{}^*$ denotes the effective mass of the exciton, and Γ_e (>0) is the phenomenological damping constant, whose dependence on $\vec{k}$ will be ignored. Equation (2) may be expressed in more compact form:

$$\hat{\epsilon}(\vec{k}, \omega) = \epsilon_0(\omega) + \chi_e(\omega)/(k^2 - \mu_e{}^2), \tag{3}$$

where

$$\chi_e(\omega) = m_e{}^*\omega_e\alpha_e/\hbar, \quad \mu_e{}^2 = (m_e{}^*/\hbar\omega_e)(\omega^2 - \omega_e{}^2 + i\omega\Gamma_e). \tag{4}$$

Let us take the three-dimensional Fourier transform on $\vec{k}$ of (1). Then on using the convolution theorem on Fourier transforms and the expression (3) for $\hat{\epsilon}(\vec{k}, \omega)$, we obtain the following relation between the Fourier frequency transforms of $\vec{E}$ and $\vec{D}$:

$$\vec{D}(\vec{r}, \omega) = \epsilon_0(\omega)\vec{E}(\vec{r}, \omega) + \frac{\chi_e}{4\pi}\int \frac{\exp(i\mu_e|\vec{r} - \vec{r}\,'|)}{|\vec{r} - \vec{r}\,'|}\vec{E}(\vec{r}\,', \omega)\, d^3r'. \tag{5}$$

Equation (5) is strictly valid only for a medium occupying the whole infinite space, but we may assume that it is valid, to a good approximation, for a medium occupying a finite but sufficiently large volume V; in that case the integration in (5) extends over the volume V only.

It readily follows from Maxwell equations, on eliminating the magnetic field and on using Eq. (5), that the electric field satisfies the following integro-differential equation:

$$\nabla \times \nabla \times \vec{E}(\vec{r}, \omega) - \frac{\omega^2}{c^2}\epsilon_0\vec{E}(\vec{r}, \omega) = \frac{\chi\omega^2}{4\pi c^2}\int_V \frac{\exp(i\mu|\vec{r} - \vec{r}\,'|)}{|\vec{r} - \vec{r}\,'|}\vec{E}(\vec{r}\,', \omega)\, d^3r', \tag{6}$$

where, for the sake of brevity, we have dropped the subscript e from μ_e and χ_e.

Equation (6) is one of the basic equations of our theory. We will now derive a general solution—in the form of a mode expansion—of this equation for the case when the volume V is the domain $0 \leq z \leq d$, $-\infty \leq x \leq \infty$, $-\infty \leq y \leq \infty$. For this purpose we take the two-dimensional Fourier transform of $\vec{E}(\vec{r}, \omega)$ with respect to the variables x and y:

$$\vec{E}(\vec{r}, \omega) = \iint_{-\infty}^{+\infty} \hat{\vec{E}}(u, v, z; \omega) e^{i(ux + vy)}\, du\, dv. \tag{7}$$

Next we recall the following representation[10] of the kernel of the integral in (6), valid if $\mathrm{Re}\mu \geq 0$, $\mathrm{Im}\mu \geq 0$:

$$\frac{\exp(i\mu|\vec{r} - \vec{r}\,'|)}{|\vec{r} - \vec{r}\,'|} = \frac{i}{2\pi}\iint_{-\infty}^{+\infty} \frac{1}{w}\exp\{iu(x - x') + iv(y - y') + iw|z - z'|\}\, du\, dv, \tag{8}$$

where

$$w = (\mu^2 - u^2 - v^2)^{1/2}, \tag{9}$$

and the square root is defined so that $\mathrm{Re}w > 0$, $\mathrm{Im}w > 0$.

On substituting from (7) and (8) into (6), we find that $\hat{\vec{E}}(u, v, z; \omega)$ satisfies the following integro-differential equation:

$$\left(u^2 + v^2 - \frac{\omega^2}{c^2}\epsilon_0\right)\hat{\vec{E}} - \frac{\partial^2}{\partial z^2}\hat{\vec{E}} + \vec{L} = \left(\frac{i\chi\omega^2}{2c^2}\right)\int_0^d \frac{e^{iw|z - z'|}}{w}\hat{\vec{E}}(u, v, z'; \omega)\, dz', \tag{10}$$

where $\vec{L}$ is the vector whose components are

$$L_x = iu\xi, \quad L_y = iv\xi, \quad L_z = \partial\xi/\partial z; \quad \xi \equiv (iu\hat{E}_x + iv\hat{E}_y + \partial\hat{E}_z/\partial z). \tag{11}$$

Equation (10) may readily be converted into the following linear differential equation with constant coefficients:

$$\frac{\partial^4\hat{\vec{E}}}{\partial z^4} + \left(\frac{\omega^2}{c^2}\epsilon_0 - u^2 - v^2 + W^2\right)\frac{\partial^2\hat{\vec{E}}}{\partial z^2} + \left\{W^2\left(\frac{\omega^2}{c^2}\epsilon_0 - u^2 - v^2\right) - \frac{\chi\omega^2}{c^2}\right\}\hat{\vec{E}} = \left(\frac{\partial^2}{\partial z^2} + w^2\right)\vec{L}. \tag{12}$$

The most general solution of (12) may be shown to be given by

$$\hat{\vec{E}}(u,v,z;\omega)=\sum_{j=1}^{4}\vec{A}_j(u,v;\omega)\exp(i\sigma_j z)+\sum_{j=1}^{2}\vec{A}_j'(u,v;\omega)\exp(i\sigma_j' z), \tag{13}$$

where σ_j and σ_j' are the roots of the equations

$$\sigma^4-\sigma^2[(\omega^2/c^2)\epsilon_0-u^2-v^2+w^2]+w^2[(\omega^2/c^2)\epsilon_0-u^2-v^2]-\chi\omega^2/c^2=0, \tag{14}$$

$$\sigma'^2-w^2+\frac{\chi}{\epsilon_0}=0. \tag{15}$$

On substituting (13) into (7), we obtain the following general expression for $\vec{E}(\vec{r},\omega)$:

$$\vec{E}(\vec{r},\omega)=\iint_{-\infty}^{+\infty}\vec{\mathcal{E}}(\vec{r},\omega;u,v)\,du\,dv, \tag{16a}$$

where

$$\vec{\mathcal{E}}(\vec{r},\omega;u,v)=\sum_{j=1}^{4}\vec{A}_j(u,v;\omega)e^{i\vec{k}_j\cdot\vec{r}}+\sum_{j=1}^{2}\vec{A}_j'(u,v;\omega)e^{i\vec{k}_j'\cdot\vec{r}}. \tag{16b}$$

In (16b), $\vec{k}_j$ and $\vec{k}'_j$ are the (complex) vectors defined by

$$\vec{k}_j\equiv(u,v,\sigma_j);\quad \vec{k}_j'\equiv(u,v,\sigma_j'). \tag{17}$$

The vector functions $\vec{A}_j$ and $\vec{A}'_j$ are, however, not quite arbitrary. They must satisfy the following constraints, that may be deduced by substituting from Eq. (13) into Eq. (10):

$$(\vec{k}_j\cdot\vec{A}_j)\vec{k}_j=0,\quad j=1,2,3,4; \tag{18}$$

$$(\vec{k}_j'\cdot\vec{k}_j')\vec{A}_j'-(\vec{k}_j'\cdot\vec{A}_j')\vec{k}_j'=0,\quad j=1,2; \tag{19}$$

$$\sum_{j=1}^{4}\frac{\vec{A}_j}{(\sigma_j-w)}+\sum_{j=1}^{2}\frac{\vec{A}_j'}{(\sigma_j'-\omega)}=0, \tag{20}$$

$$\sum_{j=1}^{4}\frac{\vec{A}_j}{(\sigma_j+w)}\exp[i(\sigma_j+w)d]+\sum_{j=1}^{2}\frac{\vec{A}_j'}{(\sigma_j'+w)}\exp[i(\sigma_j'+w)d]=0. \tag{21}$$

We now examine the nature of the solution given by Eq. (16). From the manner in which the solution was constructed, it is clear that for each u, v, and ω, $\mathcal{E}(r,\omega;u,v)$ satisfies the basic integro-differential equation (6). Thus (16a) expresses the field in the domain $0\leq z\leq d$, $-\infty\leq x\leq\infty$, $-\infty\leq y\leq\infty$ as a superposition of modes of Eq. (6), appropriate to that domain. Each mode is labeled by the parameters u, v, and ω, and is seen to be a linear superposition of six plane waves, whose wave vectors are given by Eq. (17). Since these wave vectors are, in general, complex, the waves are inhomogeneous. We set

$$\sigma_j=\alpha_j+i\beta_j,\quad \sigma_j^2=a_j+ib_j, \tag{22}$$

where α_j, β_j, a_j, and b_j are real. Then a typical plane wave of the "$\vec{A}$ type" has the spatial dependence $\exp\{i(ux+vy+\alpha_j z)-\beta_j z\}$. It is seen that the surface of constant phase of such a wave is propagated in the direction whose direction cosines are in the ratio $u{:}v{:}\alpha_j$ and the amplitude of the wave decreases or increases exponentially with increasing z according whether β_j is positive or negative. It is readily found on examining the four roots σ_j that each (u,v,ω) mode consists of two $\vec{A}$ waves and of one $\vec{A}'$ wave propagated from the plane $z=0$ towards the plane $z=d$, and of two $\vec{A}$ waves and one $\vec{A}'$ wave propagated from the plane $z=d$ towards the plane $z=0$. There is a basic difference in the $\vec{A}$ and the $\vec{A}'$ waves, as is seen at once from Eqs. (18) and (19). These equations show that the $\vec{A}$ waves are *transverse* and the $\vec{A}'$ waves are *longitudinal* (both in the generalized sense appropriate to plane waves with a complex propagation vector). If we recall that $k_j^2=u^2+v^2+\sigma_j^2$ and $k_j'^2=u^2+v^2+\sigma_j'^2$, then it is clear that Eq. (14) is the *dispersion relation for the transverse waves* and that Eq. (15) is the *dispersion relation for the longitudinal waves.*[11] Equations (20) and (21) are seen to couple these waves.

Finally, for the sake of completeness, we also write down the mode expansions for the Fourier frequency transform of the electric displacement vector $\vec{D}$ and of the magnetic fields $\vec{H}$ and $\vec{B}$. It is found on substituting from (16) into Maxwell equations and on using Eqs. (5) and (18)–(21) that

$$\vec{D}(\vec{r},\omega)=\iint_{-\infty}^{+\infty}\vec{\mathfrak{D}}(\vec{r},\omega;u,v)\,du\,dv,\quad \vec{H}(\vec{r},\omega)=\vec{B}(\vec{r},\omega)=\iint_{-\infty}^{+\infty}\vec{\mathcal{K}}(\vec{r},\omega;u,v)\,du\,dv, \tag{24}$$

where

$$\vec{\mathfrak{D}}(\vec{r},\omega;u,v)=\sum_{j=1}^{4}\left\{\epsilon_0(\omega)+\frac{\chi}{(\sigma_j^2-w^2)}\right\}\vec{A}_j(u,v;\omega)\exp(i\vec{k}_j\cdot\vec{r}), \tag{25}$$

$$\vec{\mathcal{K}}(\vec{r},\omega;u,v)=\sum_{j=1}^{4}\left(\frac{c}{\omega}\right)\{\vec{k}_j\times\vec{A}_j(u,v;\omega)\}\exp(i\vec{k}_j\cdot\vec{r}). \tag{26}$$

It is seen that no longitudinal modes appear in the expressions (25) and (26).

It is clear that our mode expansion can be used to treat a variety of problems involving the interaction of an electromagnetic field with a spatially dispersive medium. We will illustrate the technique in another paper.

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*Research supported by the U. S. Army Research Office (Durham), and the U. S. Air Force Office of Scientific Research.

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Observation of Natural-Parity Exchange in ρ^0 Production*

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(Received 12 July 1971)

In a study of $\pi^-p\to\rho^0n$ and $\pi^+n\to\rho^0p$ at 7 GeV/c we observe a distinct change in the $\pi\pi$ angular distribution for $|t_{np}|\geq 0.3$ (GeV/c)^2. Here there is a strong $\sin^2\theta\sin^2\varphi$ term in the decay angular distribution which is indicative of natural-parity exchange. The observed effects are attributed to A_2 exchange.

This paper is concerned with the production of the ρ^0 meson. We center our attention on ρ^0 production at large rather than small values of the momentum transfer. The data which we discuss were derived from extensive exposures of the Midwestern Universities Research Association–Argonne National Laboratory 30-in. bubble chamber to π^-p and π^+d at 7 GeV/c incident $\pi^\pm$ momentum.[1,2] In the course of these two exposures we have analyzed approximately 10 000 events giving rise to a π^+ and a π^- plus a recoiling nucleon ($\sim\frac{2}{3}$ of the events are from the π^+d exposure).

As is well known, at small $|t|$ values the ρ^0 production is dominated by one-pion exchange