One-Dimensional Anisotropic Heisenberg Chain

R. J. Baxter
Research School of Physical Sciences, The Australian National University, Canberra, A.C.T. 2600, Australia
(Received 25 February 1971)

A formula is given for the minimum eigenvalue of the operator of an infinitely long, one-dimensional anisotropic Heisenberg chain.

Until now the ground-state energy of the one-dimensional Heisenberg operator

$$\mathcal{H}=-\tfrac{1}{2}\sum_{J=1}^{N}(A\sigma_J{}^x\sigma_{J+1}{}^x+B\sigma_J{}^y\sigma_{J+1}{}^y+C\sigma_J{}^z\sigma_{J+1}{}^z) \tag{1}$$

has been calculated only when either one of the parameters A, B, C is zero,[1] or two are equal.[2] We sketch here the solution for arbitrary values of A, B, and C (in the limit of large N). The details of the calculation will be presented elsewhere.[3]

In a previous Letter[4] we have outlined the solution of an "eight-vertex" model in lattice statistics. It is known that a special case of this (the "ice" models[5]) is related to the Heisenberg chain with $A=B$. We might hope that the more general lattice problem is related to the general Heisenberg chain problem, and indeed we find that this is the case.

To see this we use the notation of Ref. 4, where we set up a class of commuting 2^N-by-2^N matrices $T(v)$. When $v=\eta$ it is quite easy to see that $T(v)$ is simply proportional to an operator that shifts all arrows one column to the left. Regarding k and η as constants and differentiating with respect to v, we can then deduce that

$$\frac{d}{dv}\ln T(v)\bigg|_{v=\eta}=\frac{1-k\,\mathrm{sn}^2(2\eta)}{A\,\mathrm{sn}(2\eta)}(\mathcal{H}-\tfrac{1}{2}NCE), \tag{2}$$

where E is the identity operator and $\mathcal{H}$ is given by (1) with

$$A{:}B{:}C=[1-k\,\mathrm{sn}^2(2\eta)]:[1+k\,\mathrm{sn}^2(2\eta)]:[-\mathrm{cn}(2\eta)\,\mathrm{dn}(2\eta)]. \tag{3}$$

For given values of A, B, and C we calculate k and η from (3).

From (2) it is apparent that $\mathcal{H}$ commutes with the matrices $T(v)$ and hence has the same eigenvectors. We assume (as seems reasonable from perturbation expansions) that when A is positive the eigenvector which corresponds to the maximum eigenvalue of $T(v)$ (in the principal domain of Ref. 4) also corresponds to the minimum eigenvalue $\lambda_{\min}$ of $\mathcal{H}$. Differentiating Eq. (9) of Ref. 4, we then obtain

$$F(A,B,C)=\lim_{N\to\infty}(2N)^{-1}\lambda_{\min}=\frac{C}{4}-[(C^2-A^2)^{1/2}+(C^2-B^2)^{1/2}]\frac{\pi}{2K}\sum_{n=1}^{\infty}\frac{\sinh^2[(\tau-\lambda)n]\tanh(n\lambda)}{\sinh(2n\tau)}, \tag{4}$$

where K, τ, and λ are defined in Ref. 4.

The formula (4) applies only in the "principal domain" $0<k<1$, $0<\lambda<\tau$, i.e., $|B|<A<-C$. However, $\lambda_{\min}$ is unaltered either by any interchange of A, B, and C or by negating any two of A, B, and C, so we can always ensure that this restriction is satisfied. (If we negate one or three of A, B, C, we interchange $\lambda_{\min}$ and $-\lambda_{\max}$.) We find that F is an analytic function of A, B, C inside the principal domain and across the boundaries $B=\pm A$, but across $C=-A$ it has a branch-point singularity of the type exhibited in Eq. (11) of Ref. 4, $T-T_c$ being replaced by $C+A$ and μ being given by

$$\cos\mu=B/A. \tag{5}$$

---

[1] S. Katsura, Phys. Rev. 127, 1508 (1962).
[2] C. N. Yang and C. P. Yang, Phys. Rev. 150, 321 (1966).
[3] R. J. Baxter, to be published.
[4] R. J. Baxter, preceding Letter [Phys. Rev. Lett. 26, 832 (1971)]. .
[5] E. H. Lieb, Phys. Rev. Lett. 18, 1046 (1967).