differently in the two samples. However, the low value of T_c we observe for both samples can only be attributed to error in our thermometer. We used a germanium resistance thermometer calibrated by Cryocal, Riviera Beach, Fla., and were unable to check the calibration later since the thermometer was damaged.

Eight-Vertex Model in Lattice Statistics

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The solution of the zero-field "eight-vertex" model is presented. This model includes the square lattice Ising, dimer, ice, F, and KDP models as special cases. It is found that in general the free energy has a branch-point singularity at a phase transition, with a continuously variable exponent.

It has been pointed out[1] that many of the previously solved two-dimensional lattice models, notably the Ising and "ice"-type models, can be regarded as special cases of a more general model. Adopting the arrow terminology used by Lieb,[3] we can formulate this model as follows: Place arrows on the bonds of a square, N-by-N lattice and allow only those configurations with an even number of arrows pointing into each vertex. Then there are eight possible different configurations of arrows at each vertex (hence our name for the model), as shown in Fig. 1. Next we assign energies $\epsilon_1, \cdots, \epsilon_8$ to these vertex configurations and the problem is to evaluate the partition function

$$Z = \sum \exp(-\beta \sum_{j=1}^{8} N_j \epsilon_j), \tag{1}$$

where the summation is over all allowed configurations of arrows on the lattice, and N_j is the number of vertices of type j.

Let $\omega_j = \exp(-\beta\epsilon_j)$, and suppose the model to be unchanged by reversal of all arrows (in ferroelectric terminology this implies that there are no electric fields). Then we can write

$$\omega_1 = \omega_2 = a, \quad \omega_3 = \omega_4 = b,$$
$$\omega_5 = \omega_6 = c, \quad \omega_7 = \omega_8 = d. \tag{2}$$

We have succeeded in solving this model for arbitrary values of a, b, c, and d, and outline here the main results. The details of the derivation will be published elsewhere.[3]

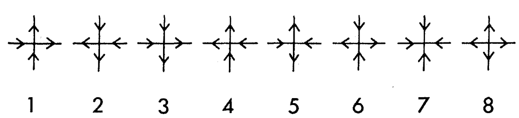

FIG. 1. The eight arrow configurations allowed at a vertex.

The method of attack on the problem was guided by some recent results[4] for an inhomogeneous system satisfying the "ice condition" ($d=0$), in which we observed that the Bethe *Ansatz* approach worked provided that the transfer matrices of any two rows commuted. This led to an algebraic identity which later reflection has shown to be the diagonal representation of an identity between the transfer matrices and another matrix function Q.

Applying these ideas to the $d \neq 0$ situation, suppose that a, b, c, and d can vary from row to row. We find that the transfer matrices for any two rows commute provided that

$$a:b:c:d = \operatorname{sn}(\eta - v) : \operatorname{sn}(\eta + v) : \operatorname{sn}(2\eta) : -k\operatorname{sn}(2\eta)\operatorname{sn}(\eta - v)\operatorname{sn}(\eta + v), \tag{3}$$

where $\operatorname{sn}(u)$ is the usual elliptic function[5] of modulus k; k and η are fixed, but v can vary from row to row. Thus we can write the transfer matrix of a typical row as $T(v)$, where v is a variable parameter.

We now look for a matrix identity of the type mentioned above and find that there exists a matrix function $Q(v)$ (in general nonsingular) such that[6]

$$\zeta(v)T(v)Q(v) = \varphi(v-\eta)Q(v+2\eta) + \varphi(v+\eta)Q(v-2\eta), \tag{4}$$

where

$$\zeta(v) = [c^{-1}H(2\eta)\Theta(v-\eta)\Theta(v+\eta)]^N, \tag{5}$$

$$\varphi(v) = [\Theta(0)H(v)\Theta(v)]^N, \tag{6}$$

$H(u)$ and $\Theta(u)$ being the elliptic theta functions[5] of modulus k. Both $T(v)$ and $Q(v)$ are 2^N-by-2^N matrices, and $Q(v)$ commutes with any matrix $T(u)$ or $Q(u)$. Thus there exists a representation (independent of v) in which the matrices in (4) are diagonal, and we can look at one such diagonal element.

From the quasiperiodic properties of $Q(v)$ we can show that it must be possible to write each of its diagonal elements in the form

$$Q(v)=\prod_{r=1}^{\frac{1}{2}N}H(v-v_r)\Theta(v-v_r) \tag{7}$$

(provided that N is even). Setting $v=v_1, \cdots, v_{N/2}$ in (4), the left-hand side vanishes and we get $N/2$ equations for $v_1, \cdots, v_{N/2}$. These can in principle be solved and the corresponding diagonal element (eigenvalue) of $T(v)$ can be calculated from (4). (These equations are analogous to the equations for the wave numbers in the Bethe *Ansatz.*) The partition function for a large lattice can then be calculated in the usual way from the maximum eigenvalue.

Let K and K' be the complete elliptic integrals[5] of the first kind of moduli k and k'. Define

$$\tau=\pi K'/2K, \quad \eta=iK\lambda/\pi, \quad v=iK\alpha/\pi. \tag{8}$$

Then we find that the free energy per vertex, f, is given by

$$-\beta f=\lim_{N\to\infty}N^{-1}\ln Z=-\beta\epsilon_5+2\sum_{n=1}^{\infty}\frac{\sinh^2[(\tau-\lambda)n][\cosh(n\lambda)-\cosh(n\alpha)]}{n\sinh(2n\tau)\cosh(n\lambda)}, \tag{9}$$

provided that k, λ, and α are real and

$$0<k<1, \quad |\alpha|<\lambda<\tau. \tag{10}$$

For given values of a, b, c, and d (constant throughout the lattice), k, η, and v (and hence τ, λ, and α) are to be calculated from (3). The restrictions (10) are equivalent to $a>0$, $b>0$, $d>0$, and $c>a+b+d$. This set of values of a, b, c, and d we call the principal domian, and we see that it corresponds to a generalization of the ordered F-model state. Fortunately we can map any set of values a, b, c, d into the principal domain (or its boundaries—as f is continuous these present no problems) by using the symmetry relations (9)-(12) of Ref. 1 (writing a, b, c, d for u_1, u_2, u_3, u_4).

Inside the principal domain f is an analytic function of a, b, c, and d. Thus as the temperature T is varied a phase transition can occur only when a, b, c, or d (or their appropriately mapped values) cross a boundary of the principal domain, and in general this will correspond to just one of a, b, d, or $c-a-b-d$ becoming zero. If two or more become zero simultaneously, we have a more complicated situation—the F and KDP models are in this category—which is not discussed here.

If the mapped values of a, b, or d become zero, we find that f is the same analytic function of T on both sides of the boundary value, so there is no phase transition. However, if $c-a-b-d$ has a simple zero at some value T_c of T, we find that f can be written as the sum of an analytic and a singular function of T. The analytic part is the same on both sides of T_c, while near T_c the singular part is proportional either to

$$\cot(\pi^2/2\mu)\,|T-T_c|^{\pi/\mu}, \tag{11a}$$

or, if $\pi/2\mu=m$ integer, to

$$2\pi^{-1}(T-T_c)^{2m}\ln|T-T_c|, \tag{11b}$$

where $0<\mu<\pi$ and

$$\cos\mu=(ab-cd)/(ab+cd). \tag{12}$$

The constant of proportionality which multiplies (11) is the same on both sides of T_c.

We conclude that if a given eight-vertex model has a phase transition, then in general the free energy has a branch-point singularity. Note that the exponent π/μ of this singularity can range continuously from one to "infinity." All previously solved cases are in a sense deceptive in that they correspond to very special values of the exponent. For instance, the Ising model and the "free-fermion" model[1] correspond to $\mu=\pi/2$, and we can think of the F model as a limiting case in which $\mu\to 0$, i.e., the singularity becomes of infinitely high order.

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[6] To derive (4) we have negated a. As Z is unaffected by this transformation, this has no effect on the subsequent equations.