Electroproduction Sum Rule*

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We present a new sum rule which relates the nucleon form factor to the integral over energy of the inelastic cross section of nucleons by charged vector currents.

Assuming local commutation relations for the fourth components of isospin currents, we may write

$$[\int e^{-i\vec{q}\cdot\vec{x}}\mathfrak{F}_{1+i2}{}^{0}(x)d^3x, \int e^{i\vec{q}\cdot\vec{y}}\mathfrak{F}_{1-i2}{}^{0}(y)d^3y]_{x_0=y_0}=2F_3, \tag{1}$$

where the $\mathfrak{F}_i{}^{\mu}(x)$ are the vector isospin currents and the F_i are the generators of isospin rotations.[1] By considering matrix elements of (1) between states in the infinite momentum frame, i.e., $p_3\to\infty$, we may derive a class of sum rules. In particular for $\vec{q}$ perpendicular to $\vec{p}=(0,0,p_3)$ for proton states the sum rule is[2]

$$[F_1{}^v(-\vec{q}^{\,2})]^2+\vec{q}^{\,2}[F_2{}^v(-\vec{q}^{\,2})]^2+\tfrac{1}{2}\vec{q}^{\,2}\int d\nu\,\nu^{-2}[\vec{q}^{\,2}B_2(\nu,-\vec{q}^{\,2})-B_4(\nu_1-\vec{q}^{\,2})]=1, \tag{2}$$

where F_1 and F_2 are nucleon isovector form factors, B_2 and B_4 are amplitudes contributing to the scattering of an imagined charged, virtual photon by the nucleon, and we exclude the contribution of the neutron intermediate state. B_2 and B_4 are defined by

$$\begin{aligned}A^{\mu\nu}(p,q)&=(2\pi)^3(p_0/M)\textstyle\sum_n\langle p|\mathfrak{F}_{1+i2}{}^{\mu}(0)|n\rangle\langle n|\mathfrak{F}_{1-i2}{}^{\nu}|p\rangle\delta^4(p+q-p_n)\\&\quad-(2\pi)^3(p_0/M)\textstyle\sum_n\langle p|\mathfrak{F}_{1-i2}{}^{\nu}(0)|n\rangle\langle n|\mathfrak{F}_{1+i2}{}^{\mu}(0)|p\rangle\delta^4(p-q-p_n),\\A^{\mu\nu}(p,q)&=p^\mu p^\nu B_1+q^\mu q^\nu B_2+\tfrac{1}{2}(p^\mu q^\nu+p^\nu q^\mu)B_3+g^{\mu\nu}B_4.\end{aligned} \tag{3}$$

By taking the derivative of Eq. (2) with respect to q^2, at $q^2=0$, one then obtains the well-known Cabibbo-Radicati sum rule.[3]

We wish to consider the limit of (1) between proton states with $p_3\to\infty$ and now, rather than $\vec{q}=(\vec{q}_\perp,0)$, we take $\vec{q}=(0,0,q_3)$ with $q_3\to\infty$. We further demand that the ratio q_3/p_3 remain fixed as $q_3,p_3\to\infty$ and call this ratio λ. At first sight it might appear that the momentum transfer between states is infinite, but this is not so; it is given by

$$q^2=(E_n-E)^2-(\vec{p}_n-\vec{p})^2=\{[(p_3\pm q_3)^2+M_n{}^2]^{1/2}-(p_3{}^2+M^2)^{1/2}\}^2-q_3{}^2\xrightarrow[\substack{p_3,q_3\to\infty,\\ q_3/p_3=\lambda}]{}-\frac{\lambda}{1\pm\lambda}[\lambda M^2\pm(M^2-M_n{}^2)], \tag{4}$$

for an intermediate state of mass M_n. The feature of finite momentum transfer becomes clearer if we transform to the rest frame the commutator under consideration[4]:

$$\begin{aligned}&\lim_{\substack{p_3,q_3\to\infty,\\ q_3/p_3=\lambda}}\int e^{-iq_3x_3}d^3x_\perp dx_3\langle p_3\to\infty|[\mathfrak{F}_+{}^0(0,\vec{x}),\mathfrak{F}_-{}^0(0)]|p_3\to\infty\rangle\\&=\gamma\int e^{-iq_3x_3/\gamma}d^3x_\perp dx_3\langle\vec{p}=0|[\eta\cdot\mathfrak{F}_+(-x_3;x_\perp,x_3),\eta\cdot\mathfrak{F}_-(0)]|\vec{p}=0\rangle\\&=\gamma\int e^{-iM\lambda x_3}d^3x_\perp dx_3\langle\vec{p}=0|[\eta\cdot\mathfrak{F}_+(-x_3;x_\perp,x_3),\eta\cdot\mathfrak{F}_-(0)]|\vec{p}=0\rangle,\\&\eta_\mu=(1;00-1),\quad\gamma=p_0/M,\end{aligned} \tag{5}$$

which shows that q^2 is finite; the result for q^2 of course agrees with our previous expression.

We have also verified that (1) is a so-called good commutation relation in the $p_3,q_3\to\infty$, q_3/p_3 fixed limit, in the sense that its matrix elements between free quark states are saturated by the single free-quark intermediate state.[2] We then proceed to examine the matrix element of (1) between proton states: We find, separating out the contribution of the intermediate neutron state, that

$$[F_1{}^v(\Delta^2)+\Delta^2F_2{}^v(\Delta^2)/2M]^2+(M/p_0)\int_{-\infty}^{\infty}dq_0A^{00}(q,p)=1,\quad\Delta^2=-\lambda^2M^2/(1+\lambda), \tag{6}$$

where Δ^2 is the value of q^2 corresponding to $M_n=M$, the nucleon mass. The integral can be rewritten covariantly by introducing the variable $\nu = q\cdot p$. Since $\vec{q} = (0,0,q_3)$ is fixed we have

$$q_0 = (\nu + \vec{q}\cdot\vec{p})/p_0 = (\nu + q_3 p_3)/p_0, \quad dq_0 = d\nu/p_0, \quad q^2 = q_0^2 - q_3^2 \xrightarrow[p_3\to\infty]{} 2\lambda\nu - \lambda^2 M^2, \tag{7}$$

and our sum rule becomes, as $p_3 \to \infty$,

$$\left[F_1^{\,v}(\Delta^2) + \frac{\Delta^2 F_2^{\,v}(\Delta^2)}{2M}\right]^2 + M\int_{-\infty}^{\infty} d\nu\, [B_1(\nu,q^2) + \lambda^2 B_2(\nu,q^2) + \lambda B_3(\nu,q^2)]_{q^2 = 2\lambda\nu - \lambda^2 M^2} = 1. \tag{8}$$

Labeling the B's in (8) with a superscript, $B \to B^-$, to indicate that it denotes scattering by negative currents, we can relate by crossing $B^-(-\nu,q^2)$ to $B^+(\nu,q^2)$, the scattering by positive currents, and find as always, with the one neutron state separated out from the integral, that

$$\left[F_1^{\,v}(\Delta^2) + \frac{\Delta^2 F_2^{\,v}(\Delta^2)}{2M}\right]^2 + M\int_0^{\infty} d\nu \{B_2^-(\nu,q_-^2) - B_1^+(\nu,q_+^2) + \lambda^2[B_2^-(\nu,q_-^2) - B_2^+(\nu,q_+^2)]$$
$$+ \lambda[B_3^-(\nu,q_-^2) + B_3^+(\nu,q_+^2)]\} = 1, \quad q_\pm^2 = \mp 2\lambda\nu - \lambda^2 M^2, \quad 0 \le |\lambda| < 1. \tag{9}$$

By current conservation, only two of the B's are independent, so we may express them in terms of $W_1^\pm$ and $W_2^\pm$, the charged analogs of the scalars W_1 and W_2 measured in inelastic electron-proton scattering.[5]

The sum rule, given in (9), has several interesting features: It is unphysical in that there are no charged photons, but one could estimate resonance contributions to it. We plan, in a future publication, to give such an evaluation, as well as to present the analogous sum rule for neutrino-proton inelastic scattering. Note that B^- and B^+ are evaluated at different values of q^2; as $\nu/M^2 \gg 1$ we enter into the scaling limit,[5] namely, the ratio $q^2/2\nu$ is fixed and we must demand that the B's have well-defined limits at q^2, $\nu \to \infty$, $q^2/2\nu = \pm\lambda$, and be such that the integral converges. One particularly interesting aspect of the sum rule is that q^2 ranges over positive values; in particular we might expect B^- to become large when q^2 is close to the location of the ρ-meson pole. It is amusing to note that increasing λ makes the neutron contribution smaller, but can make the contribution of B^- larger by enhancing the ρ-pole's effect, which would enter at a smaller ν, as we increase λ.

To summarize, we have derived a meaningful continuous class of sum rules, functions of a parameter λ that ranges in magnitude from 0 to 1 (as $\lambda \to 0$, which is the same as $q_3 \to 0$, the integral vanishes and we have identity,[6] whereas for $\lambda \to 1$, we are no longer in the infinite momentum frame for the crossed term so our assumptions break down). Rather than being characterized by fixed $\vec{q}^{\,2}$, our sum rule has q^2 linearly related to ν with λ the parameter characterizing the relation.

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[1]Our notation is that of S. Adler and R. Dashen, *Current Algebras and Applications to Particle Physics* (Benjamin, New York, 1968).

[2]See Ref. 1, Chap. 4.

[3]N. Cabibbo and L. Radicati, Phys. Lett. 12, 697 (1965).

[4]K. Bardakci and G. Segrè, Phys. Rev. 153, 1263 (1967); L. Susskind, Phys. Rev. 165, 1535 (1967); H. Leutwyler, in *Springer Tracts in Modern Physics*, edited by G. Höhler (Springer, Berlin, 1969), Vol. 50.

[5]J. D. Bjorken, Phys. Rev. 179, 1547 (1969). Note a difference of P_0/M between this reference's definition of the scalars in Eq. (3), which we follow, and those of Ref. 1. This corresponds to a different normalization of states.

[6]B_1 vanishes for kinematical reasons as $\lambda \to 0$.