Anomalous Viscosity as a Possible Explanation for an Anomalous Plasma Skin Effect

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Anomalous viscosity of electrons across the magnetic field in plasmas is calculated, assuming that the electrostatic ion acoustic wave is excited by the drift velocity of electrons parallel to the magnetic field lines. The mean free path of the wave perpendicular to the magnetic field is estimated. The momentum transfer rate between waves and electrons, τ^{-1}, also was calculated from the quasilinear theory. The anomalous (kinematic) viscosity was then estimated as $l^2\tau^{-1}$. The theory appears to explain the experimentally observed results of the anomalous skin effect.

It has remained a mystery why there appears to be no detectable skin effect in Tokamaks; that is, the laser electron-temperature–profile measurement failed to show the increase in the temperature at the periphery. This obviously can be explained in two ways. One is that there is an enhanced heat conductivity to the center which makes the temperature profile almost flat, in spite of the fact that current distribution is restricted to the surface. The second is that there is an anomalous skin effect which allows the current to penetrate to the center. Although this was not experimentally checked in Tokamak devices, it is known that the θ-pinch and collisionless shock waves have an anomalous skin effect. Here, we propose that an anomalous viscosity may give rise to the anomalous skin effect. This also implies that the anomalous skin effect need not be equivalent to anomalous resistivity. Experimentally, the resistivity found after the Tokamak plasmas are established cannot explain the seemingly rapid penetration of the current across the magnetic field. This concept of anomalous viscosity was introduced by Morse and Stovall[1] to explain the observed field reversal in θ pinches.

The physical model we propose here is very simple. We assume that the plasma density n_0 is uniform, that the electron temperature T_e is constant, and that the ion temperature is negligible. The current density, which is carried by electrons, has a gradient in the x direction and a strong, constant magnetic field in the z direction. The ion-plasma frequency ω_{pi} is considered to be much higher than the ion cyclotron frequency Ω_i. The system is unstable against many modes, such as the two-stream instability and ion acoustic waves. We assume that the nonlinear limit has set in. Thus, the plasma no longer is quiescent, but has a finite density of phonons. Since ion temperature is assumed to be cold, no strong interaction appears between phonons and ions, although the interaction may play a part in setting the nonlinear limitation of phonon density. The electrons then interact with the phonons and exchange momentum. The momentum is then carried by phonons across the magnetic field and imparted to the electrons whose parallel velocity is smaller than the phase velocity of the phonons. In this way the momentum is carried out from one electron to another electron via phonons. Therefore, if we know the density of the phonons, n_p (m^{-3}), the collision rate between a phonon and an electron, S (m^3/sec), and the mean free path of the phonon perpendicular to the magnetic field, l (m), then the kinematic viscosity perpendicular to B is given by n_pSl^2. We now try to estimate S and l. Unfortunately, n_p cannot be estimated unless the full nonlinear equation is solved. Instead, as will be explained, we shall express the

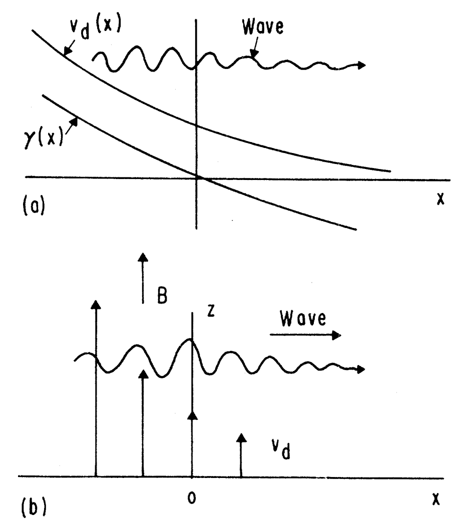

FIG. 1. (a) Drift velocity v_d and growth rate γ as a function of x (perpendicular to the magnetic field). (b) Wave propagation, $+x$ direction and absorbed for $x > 0$. The momentum parallel to B is transferred from a point $x < 0$ to another point, $x > 0$.

phonon density in terms of the plasma density fluctuation.

In practice, instead of phonons we calculate the wave-electron interaction, which is the classical picture of the phonon interaction.

As waves that have perpendicular group velocity, we chose electrostatic ion cyclotron waves.[2] For the wave with an $\exp[i(\omega t+k_x x+k_z z)]$ dependence,[3] the dispersion relation is

$$\lambda_D^2 k_x^2\left[1+\frac{\omega_{pi}^2}{\Omega_i^2-\omega^2}\right]+k_z^2\lambda_D^2\left(1-\frac{\omega_{pi}^2}{\omega^2}\right)+1=0. \quad (1)$$

Here, λ_D is the Debye length of electrons. We look for the solution for which the electron drift frequency is of the order of $|\omega/k_z|$. Then, provided that the drift frequency is higher than the plasma sound velocity (which we assume),

$$k_x^2\approx\frac{\omega^2-\Omega_i^2}{\Omega_i^2+\omega_{pi}^2-\omega^2}\frac{\omega_{pi}^2}{v_s^2},\quad v_s^2\equiv\frac{kT_e}{M}. \quad (2)$$

Waves propagate in this frequency range: $\Omega_i^2<\omega^2<\omega_{pi}^2$. We restrict to ourselves waves for which $\omega^2\ll\omega_{pi}^2$. Then,

$$k_x^2\approx\frac{\omega^2-\Omega_i^2}{v_s^2}\equiv k_0^2\quad(k_0>0). \quad (3)$$

The growth (or damping) rate of this wave can be easily calculated from the standard Landau damping calculation. We take the zeroth-order distribution function as a displaced Maxwellian. The result is

$$k_x^2 v_s^2=(\omega^2-\Omega_i^2)\left[1+\left(-\frac{\omega}{k_z}-v_d\right)\pi i f_{0e}\left(-\frac{\omega}{k_z}\right)\frac{1}{n_0}\frac{k_z}{|k_z|}\right]; \quad (4)$$

here f_{0e} is the zeroth-order electron distribution function. The growth rate γ is given by [for $|\omega/k_z|\ll(kT_e/m)^{1/2}$]

$$\gamma=\frac{k_x^2}{2|k_z|}\frac{(kT_e m)^{1/2}}{M}\left(\frac{\pi}{2}\right)^{1/2}\left(-\frac{v_d k_z}{\omega}-1\right). \quad (5)$$

Thus, if $v_d k_z/\omega<-1$, the plasma is unstable.

We first estimate the mean free path of the wave perpendicular to the magnetic field. There, noting that v_d is a function of x (Fig. 1), we take $\varphi=\varphi(x)\exp[i(\omega\tau+k_z z)]$. Then we write, instead of Eq. (4) (we take $k_z>0$),

$$\frac{d^2\varphi(x)}{dx^2}+\frac{\omega^2-\Omega_i^2}{v_s^2}\left[1+\left(-\frac{\omega}{k_z}-v_d\right)\left(\frac{\pi}{2}\frac{m}{kT_e}\right)^{1/2}i\right]\varphi(x)=0. \quad (6)$$

Assuming that the second term in the bracket is small, we apply the eikonal approximation, obtaining

$$\varphi(x)=\exp(ik_0x)\exp\left[-\int\left(-\frac{\omega}{k_z}-v_d\right)\left(\frac{\pi}{2}\frac{m}{kT_e}\right)^{1/2}\frac{dx\,k_0}{2}\right]. \quad (7)$$

In order to estimate the integral, we choose the coordinates so that $v_d=v_{d1}-(x/L)v_{d0}$, with v_{d1} and L positive. We expect that v_{d0} and v_{d1} are comparable to the typical drift velocity of electrons. Near $x=0$, $v_d>0$, so only the $\omega/k_z<0$ wave is unstable. We calculate the penetration depth for the phase velocity $\omega/k_z=-v_{d1}$. Then $(-\omega/k_z-v_d)=v_{d0}x/L$. Equation (7) then becomes

$$\varphi(x)=\exp(ik_0x)\exp\left[-\frac{1}{4}\frac{k_0x^2v_{d0}}{L}\left(\frac{\pi}{2}\frac{m}{kT_e}\right)^{1/2}\right],\quad x\geqslant0. \quad (8)$$

The depth l for which the amplitude decreases by a factor $e^{-1/2}$ is

$$l^2=\left(\frac{2kT_e}{\pi m}\right)^{1/2}\frac{1}{v_{d0}}\frac{2L}{k_0},\quad v_s^2k_0^2=\omega^2-\Omega_i^2. \quad (9)$$

The interaction of the wave with the electron distribution function can be estimated from the quasilinear theory.[4,5] In the absence of the external electric field and collision, and if $\vec{E}$ is not too large (for more detail, see Sec. 6 of Ref. 5),

$$\frac{\partial f_2}{\partial t}=\frac{1}{2}\mathrm{Re}\left(\frac{\partial f_1}{\partial v_z}E_z^*\right)=\frac{1}{2}\frac{e^2}{mkT_e}\mathrm{Re}\left\{E_z^*\varphi\frac{\partial}{\partial v_z}\left[\frac{(-\omega-k_zv_d)}{\omega+k_zv}f_{0e}(v)\right]\right\}, \quad (10)$$

or

$$\frac{\partial}{\partial t}\Gamma \equiv \frac{\partial}{\partial t}\int v_z f_2 dv = \frac{1}{2}\frac{e^2}{mkT_e}|\varphi|^2\left(\frac{\pi m}{2kT_e}\right)^{1/2}(-\omega - k_z v_d)n_0\frac{k_z}{|k_z|} = \frac{1}{2}\frac{e^2}{mkT_e}|\varphi|^2\left(\frac{\pi m}{2kT_e}\right)^{1/2}\left(-\frac{\omega}{k_z v_d}-1\right)\Gamma_0|k_z|, \quad (11)$$

with $\Gamma_0 \equiv \int v_z f_0 dv$. Thus, noting that $\gamma > 0$ corresponds to

$$-\frac{\omega}{k_z v_d} - 1 < 0, \quad (12)$$

we find that Γ changes by Γ_0 in the time τ, with

$$\tau^{-1} = \frac{1}{2}\left(\frac{e^2}{mkT_e}\right)|\varphi|^2\left(\frac{\pi m}{2kT_e}\right)^{1/2}\left(1 + \frac{\omega}{k_z v_d}\right)|k_z|. \quad (13)$$

Therefore, the kinematic viscosity is estimated to be

$$\mu = \frac{l^2}{\tau} = \sum \frac{2kT_e}{\pi m}\frac{1}{v_{d0}}\frac{L}{k_0}\left|\frac{e\varphi}{kT_e}\right|^2\left(1 + \frac{\omega}{k_z v_d}\right)|k_z|, \quad (14)$$

where the summation sign indicates the sum over all the modes. We may rewrite Eq. (14) as

$$\mu \approx \frac{2kT_e}{\pi m}\frac{L}{v_{d0}^2}\sum\left|\frac{k_z v_{d0}}{\omega}\right|\left|\frac{e\varphi}{kT_e}\right|^2\frac{\omega}{(\omega^2-\Omega_i^2)^{1/2}}\left(\frac{kT_e}{M}\right)^{1/2} = 2L\left(\frac{kT_e}{M}\right)^{1/2}\frac{1}{\pi}\frac{kT_e}{mv_{d0}^2}\left\langle\left|\frac{k_z v_{d0}}{\omega}\right|\frac{\omega}{(\omega^2-\Omega_i^2)^{1/2}}\right\rangle_{\rm av}\sum\left(\frac{e\varphi}{kT_e}\right)^2, \quad (15)$$

where the average is taken over all the modes. This average is taken of the order of unity and denoted by C. We also note that $e\varphi/kT_e \approx \tilde{n}/n_0$ from the dispersion relation where $\tilde{n}$ is the density perturbation. Hence,

$$\mu = 2L\left(\frac{kT_e}{M}\right)^{1/2}\frac{1}{\pi}\frac{kT_e}{mv_{d0}^2}C\left\langle\frac{(n-n_0)^2}{n_0^2}\right\rangle_{\rm av}, \quad (16)$$

with the assumption that the $(n-n_0)^2$ average is taken over the frequency region between Ω_i and ω_{pi}. L is related to the drift velocity density gradient, and may be approximated as $1/L = |d(\ln v_d)/dx|$.

With this kinematic viscosity given, it is possible to estimate the skin effect. Equations to govern the skin effect are given by

$$(\nabla \times \vec{B})_z = \mu_0 J_z = -e\mu_0\Gamma_z, \quad (17)$$

and (ν_C is the Coulomb collision frequency)

$$en_0 E_z = +m\mu\nabla^2\Gamma_z - m\nu_C\Gamma_z - m\,\partial\Gamma_z/\partial t. \quad (18)$$

Introducing vector potential $\vec{A}$, and applied constant electric field E_0 in the z direction, we get

$$(m\mu\nabla^2 - m\nu_C)\nabla^2 A_z = m\nabla^2\frac{\partial A_z}{\partial t} - e^2\mu_0 n_0\frac{\partial A_z}{\partial t} + e^2\mu_0 n_0 E_0. \quad (19)$$

We now determine the time constant for a slab with the thickness of $2x_0$. (Asymptotically, the solution is $A_z = -x^2e^2\mu_0 n_0 E_0/2m\nu_C$.) The normal-mode solution for the lowest eigenmode is then

$$A_z = A_0\cos\left(\frac{\pi}{2}\frac{x}{x_0}\right)\exp\left(\frac{-t}{\tau_s}\right), \quad (20)$$

$$\frac{1}{\tau_s} = \frac{1}{e^2\mu_0 n_0}\frac{\pi^2}{4}\frac{m}{x_0^2}\left(\frac{\pi^2}{4}\frac{1}{x_0^2}\mu + \nu_C\right)\frac{1}{1+\pi^2c^2/4x_0^2\omega_{pe}^2}. \quad (21)$$

Here, $\omega_{pe}/2\pi$ is the electron plasma frequency. The gradient distance L is estimated to be $2x_0/\pi$. Then

$$\tau_s = \frac{4e^2\mu_0 n_0 x_0^2}{\pi^2 m}\left[\frac{C}{x_0}\left(\frac{kT_e}{M}\right)^{1/2}\frac{kT_e}{mv_{d0}^2}\left\langle\frac{(n-n_0)^2}{n_0^2}\right\rangle + \nu_C\right]^{-1}\left(1+\frac{\pi^2c^2}{4x_0^2\omega_{pe}^2}\right). \quad (22)$$

Now the estimate of the new term is in order. For the Tokamak,[6] we assume that $\langle(n-n_0)^2/n_0^2\rangle = 10^{-2}$, $x_0 = 10$ cm, $C = 1$, $M/m = 1800$, $n_0 = 2\times10^{13}$ cm^{-3}, $kT_e/mv_{d0}^2 = 10^2$; we then have the figures shown in Table I.

In the table, τ_s* is the skin time without the viscosity correction.

It thus appears that the anomalous skin effect, as observed in Tokamaks, may be traceable to this effect. An experimental check should be relatively easy, especially if the time of current penetration is measured after the plasma is heated to a relatively high temperature. Oscillation over the ion cyclotron frequency should also be measurable.

Somewhat paradoxical results are indicated in Eq. (22), because it may appear that increasing the drift velocity increases τ_s. However, in reality $\langle (n-n_0)^2/n_0^2\rangle_{av}$ should be a function of v_{d0}, and presumably an increasing function. This probably compensates for the inverse-square dependence on v_{d0}.

Aside from the application to the Tokamak, this mechanism may also be responsible for the anomalously thick skin of collisionless shocks perpendicular to the magnetic field, as well as the anomalous viscosity hypothesized by Morse and Stovall.[1]

Furthermore, we have introduced here a new concept for the mechanism of transport of various plasma parameters. This technique may be useful for calculating heat conductivity and diffusion coefficients.

In the absence of both this viscosity effect and ion Landau damping, the electron and wave momenta would reach equilibrium values such as predicted by Drummond and Pines[7,8] and the resistivity due to the phonon-electron interaction would disappear. Thus either ion Landau damping (which also leads to anomalous resistivity) or anomalous viscosity is essential in explaining the anomalous skin effect.

Table I. Skin times calculated for different electron temperatures.

kT_e (eV)	τ_s (msec)	τ_s* (msec)
100	2.1	7.2
400	1.45	57.6
1000	0.95	220
2000	0.65	640

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Comment on the Proposed Magnetic Cooling of He^3

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We examine a recent proposal by Goldstein for producing very low temperatures by freezing He^3 in the presence of magnetic fields. We conclude that the highly isotropic nature of solid He^3 will completely suppress the particular cooling mechanism envisaged, but a related magnetothermal effect should be observable. At readily accessible temperatures this effect will be small, but below about 1 mK a potentially useful amount of magnetic cooling is predicted. The field required for the maximum cooling is about 70 kOe.

In a recent Letter,[1] Goldstein proposed a new method for obtaining extremely low temperatures by freezing He^3 in the presence of a magnetic field. It was implied that the method depends on the peculiar nature of the anisotropy of solid He^3, but the precise role of the anisotropy was not discussed explicitly.[2] In this note we wish to point out that (1) the particular mean-field calcu-