Singular Core Interactions and Three-Body Theory*

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It is shown that the Faddeev formalism must and can be modified when the short-range two-body behavior is specified by a hard core or boundary-condition model. A new integral equation is formulated for this purpose which, in the special case of boundary-condition model or hard core alone, reduces to an equation in a single vector variable. This feature provides an apparently unique example of a nonseparable interaction for which the three-body equations can be solved exactly.

There are many problems in physics in which it is a useful abstraction to represent the extremely short-range interaction of a two-particle system by introducing a hard core or its generalization, the boundary-condition model (BCM). A test of such ideas and many interesting applications lie, potentially, in systems consisting of three or more particles. In order to realize this potential one must first learn how properly to incorporate such singular interactions into the existing three-particle formalism. In this Letter I present one solution to this problem.

A recent paper[1] by the present author demonstrated that it is possible to define unambiguously an off-shell two-body t matrix appropriate to singular core interactions of the above type. In what follows I employ some special properties of this t matrix to show that the usual Faddeev equations[2] do not have a unique solution for a potential model consisting of the BCM[3] and an arbitrary potential external to the core. It is further demonstrated that this ambiguity can be eliminated by an alternative interpretation of the three-body formalism. For the special case of BCM alone, it is shown that the resultant problem reduces to the solution of an integral equation in a single vector variable. If, in addition, one makes the assumption that only a few two-body partial waves contribute (which is quite reasonable for a model with finite range), the problem may be further reduced to a set of coupled one-dimensional equations, and hence is exactly soluble.[4] To the author's knowledge, this is the only example of a *nonseparable* interaction for which such a reduction is possible.

I emphasize that the above statements hold regardless of whether the BCM t matrix is diagonal in l, i.e., whether or not tensor mixing is put into the model; this permits a straightforward application of the Feshbach-Lomon model[5] to the three-nucleon problem. For the more realistic case of BCM plus external potential, the corresponding statement is that the increase in computational difficulty involved in the addition of the BCM to a given external potential is equivalent to that produced by adding a separable potential.

We proceed by establishing some notation[6] for our three-body formalism, assuming for simplicity that our three particles are spinless (it should be clear from what follows that this is nonessential). We denote the mass of particle α by m_α, and the total three-body c.m. energy by W. Three-particle states are described by the usual Jacobi variables $\vec{p}_\alpha, \vec{q}_\alpha$, with the corresponding reduced masses μ_α, M_α, where

$$\mu_\alpha^{-1} = m_\beta^{-1} + m_\gamma^{-1},$$

$$M_\alpha^{-1} = m_\alpha^{-1} + (m_\beta + m_\gamma)^{-1}. \tag{1}$$

In the usual channel decomposition, the three-body state vector is $|\Psi\rangle = \sum_\alpha |\psi_\alpha\rangle$, where

$$|\psi_\alpha\rangle = (1 - G_0 t_\alpha)|\varphi\rangle - G_0 t_\alpha \sum_{\beta\neq\alpha} |\psi_\beta\rangle. \tag{2}$$

Here t_α represents the two-body t matrix as an operator in the three-body Hilbert space, $|\varphi\rangle$ is a plane-wave state, and $G_0 = G_0(W)$ is the free Green function. Equation (2) is one expression of the Faddeev equations.

It is convenient to introduce the states $|\alpha\vec{p}\vec{q}\rangle$, where

$$\langle\alpha\vec{p}\,'\vec{q}\,'|\beta\vec{p}\vec{q}\rangle = \delta_{\alpha\beta}\delta(\vec{p}\,' - \vec{p})\delta(\vec{q}\,' - \vec{q}),$$

$$\sum_\alpha \int d\vec{p}\,d\vec{q}\,|\alpha\vec{p}\vec{q}\rangle\langle\alpha\vec{p}\vec{q}| = 1. \tag{3}$$

We can then define the operators t, I, such that

$$\langle\alpha\vec{p}'\vec{q}'|t|\beta\vec{p}\vec{q}\rangle = \delta_{\alpha\beta}\delta(\vec{q}'-\vec{q})t_\alpha(\vec{p}',\vec{p};W-q^2/2M_\alpha),$$

$$\langle\alpha\vec{p}'\vec{q}'|I|\beta\vec{p}\vec{q}\rangle = -\delta\left(\vec{p}+\frac{\mu_\beta}{m_\gamma}\vec{p}'-\frac{\mu_\beta}{M_\alpha}\vec{q}'\right)\delta\left(\vec{q}+\vec{p}'+\frac{\mu_\alpha}{m_\gamma}\vec{q}'\right)\ (\alpha\beta\gamma\text{ cyclic}),$$

$$= -\delta\left(\vec{p}+\frac{\mu_\beta}{m_\gamma}\vec{p}'+\frac{\mu_\beta}{M_\alpha}\vec{q}'\right)\delta\left(\vec{q}-\vec{p}'+\frac{\mu_\alpha}{m_\gamma}\vec{q}'\right)\ (\beta\alpha\gamma\text{ cyclic}). \tag{4}$$

Here $t_\alpha(\vec{p}',\vec{p};s)$ is the two-body off-shell t matrix for particles β and γ, energy s; the diagonal elements of I vanish. With the identification

$$\psi_\alpha(\vec{p}_\alpha,\vec{q}_\alpha) = \langle\vec{p}_\alpha\vec{q}_\alpha|\psi_\alpha\rangle \equiv \langle\alpha\vec{p}_\alpha\vec{q}_\alpha|\psi\rangle, \tag{5}$$

and letting $|\psi\rangle = M|\varphi\rangle$, we can rewrite Eq. (2) in the form

$$M = 1 - G_0t + G_0tIM. \tag{6}$$

It is important to keep in mind that the operators in Eq. (6) act on the states of Eq. (3); in particular

$$\langle\alpha\vec{p}'\vec{q}'|G_0|\beta\vec{p}\vec{q}\rangle = \frac{\delta_{\alpha\beta}\delta(\vec{p}'-\vec{p})\delta(\vec{q}'-\vec{q})}{p^2/2\mu_\alpha+q^2/2M_\alpha-W-i\epsilon}. \tag{7}$$

Clearly, I and G_0 commute.

The development up to this point has been completely general, with the object of obtaining the operator equation for M, Eq. (6), as a representation of the Faddeev equations. We now observe a special property of the BCM t matrix which is a simple consequence of the explicit form given in I,

$$\tilde{V}G_0t = tG_0\tilde{V} = \tilde{V}. \tag{8}$$

As in I, $\tilde{V} = \tilde{V}\tilde{V}$ corresponds to a square-well potential of unit strength and a range a_α for the matrix element

$$\langle\alpha\vec{p}'\vec{q}'|\tilde{V}|\beta\vec{p}\vec{q}\rangle = \delta_{\alpha\beta}\delta(\vec{q}'-\vec{q})\tilde{V}_\alpha(\vec{p}'-\vec{p}). \tag{9}$$

Note that $\tilde{V}$ is *not* the potential which gives rise to the BCM t matrix. Equation (8) is valid whether or not an external potential is present.

The proof that Eq. (6) does not yield a unique solution when the BCM is present in the interaction rests on the validity of Eq. (8), and the existence of an operator B such that

$$\tilde{V}(1-I)\tilde{V}B(1-I) = \tilde{V}(1-I). \tag{10}$$

An explicit form for B is easily constructed by considering this equation in coordinate space, in which B is simply a product of θ functions. Its momentum-space representation is given by

$$\langle\alpha\vec{p}'\vec{q}'|B|\beta\vec{p}\vec{q}\rangle = \delta_{\alpha\beta}\{\delta(\Delta\vec{p})\delta(\Delta\vec{q}) - \tfrac{1}{2}\delta(\vec{p}_1)\tilde{V}_\sigma(\vec{p}_2) - \tfrac{1}{2}\delta(\vec{p}_2)\tilde{V}_\epsilon(\vec{p}_1) + \tfrac{1}{3}\tilde{V}_\epsilon(\vec{p}_1)\tilde{V}_\sigma(\vec{p}_2)\},$$

where

$$\Delta\vec{p} = \vec{p}-\vec{p}',\quad \Delta\vec{q} = \vec{q}-\vec{q}',\quad \vec{p}_1 = \Delta\vec{p} + (\mu_\alpha/m_\epsilon)\Delta\vec{q},$$
$$\vec{p}_2 = -\Delta\vec{p} + (\mu_\alpha/m_\sigma)\Delta\vec{q}, \tag{11}$$

and $\alpha\sigma\epsilon$ are cyclic.

Given B, we define the operator

$$Q = 1 + \tilde{V}B(I-1), \tag{12}$$

which has the following properties:

$$QQ = Q,\quad \tilde{V}(1-I)Q = (1-I)Q\tilde{V} = 0,$$
$$(1-\tilde{V}I)Q = 1\tilde{V},\quad Q\tilde{V} = \tilde{V}Q\tilde{V}. \tag{13}$$

It is easy to verify that I has an inverse, in fact $I^{-1} = (I+1)/2$. The properties of Q then imply that

$$(1-G_0tI)I^{-1}G_0Q\tilde{V} = G_0(1-tG_0)\tilde{V}Q\tilde{V} = 0, \tag{14}$$

where we have used Eq. (8). Thus $I^{-1}G_0Q\tilde{V} = G_0Q\tilde{V}$ is a nontrivial solution of the homogeneous equation related to Eq. (6). Consequently, a unique solution M to Eq. (6) does not exist, i.e., the inverse operator $(1-G_0tI)^{-1}$ does not exist for the BCM plus any external potential.

A better understanding of this difficulty and a key to its solution can be obtained by considering the implications of Eq. (8) for the two- and three-body wave functions. In the two-body case, the scattering state can be expressed as

$$|\psi^{(2)}\rangle=(1-G_0t)|\varphi\rangle. \tag{15}$$

Thus, as a consequence of Eq. (8), $\tilde{V}|\psi^{(2)}\rangle=0$. Physically, this simply reflects the fact that the wave function vanishes when the two particles are closer than their core radius. Moreover, this property implies that we can write

$$1-G_0t=(1-\tilde{V})(1-G_0\tilde{t}), \tag{16}$$

where $\tilde{t}$ is an operator such that $(1-G_0\tilde{t})|\varphi\rangle$ gives the correct form of $|\psi^{(2)}\rangle$ exterior to the core region. In fact, it is easy to show from the results of I that $\tilde{t}$ can be chosen to be

$$\tilde{t}=\tilde{t}^{\,\mathrm{BC}}+(1-\tilde{t}^{\,\mathrm{BC}}G_0)V_e(1-G_0t), \tag{17}$$

where V_e is the external potential, and $\tilde{t}^{\,\mathrm{BC}}$ is the half-on-shell t matrix for the pure BCM,

$$\tilde{t}^{\,\mathrm{BC}}(\vec{p}\,',\vec{p};K^2)=t^{\mathrm{BC}}(\vec{K},\vec{p};K^2). \tag{18}$$

Furthermore, $\tilde{t}$ satisfies the unitarity relation

$$\Delta\tilde{t}=-\tilde{t}^{\,+}\Delta G_0\tilde{t}^{\,-}$$
$$=-\tilde{t}^{\,-}\Delta G_0\tilde{t}^{\,+}, \tag{19}$$

where $\Delta\tilde{t}$ and ΔG_0 are the respective right-hand discontinuities of $\tilde{t}$ and G_0.

In the three-body case we have $|\Psi\rangle=(1-I)|\psi\rangle$ in our notation, and hence Eqs. (6) and (8) imply that

$$\tilde{V}|\Psi\rangle=\tilde{V}(1-I)M|\varphi\rangle=0. \tag{20}$$

In a fashion analogous to the above, Eq. (20) merely states that the three-body wave function vanishes when any pair of particles are closer than their respective core radius. Recalling the properties of our operator Q, we may extend the analogy by writing $M=QM^{\mathrm{ext}}$; Eq. (6) then suggests that

$$M^{\mathrm{ext}}=1-G_0\tilde{t}+G_0\tilde{t}IQM^{\mathrm{ext}}. \tag{21}$$

Employing Eqs. (13) and (16), we note that

$$(1-\tilde{V}I)M=(1-\tilde{V})M^{\mathrm{ext}}$$
$$=1-G_0t+(G_0t-\tilde{V})IM, \tag{22}$$

verifying that M is indeed a solution to Eq. (6). Moreover, using the relation

$$(1-I)M=1-G_0T \tag{23}$$

between M and the three-body t matrix T, as well as Eqs. (21) and (19), one can directly verify that T satisfies three-body unitarity providing that M^{ext} is a solution to Eq. (21).

The existence of M^{ext} may be demonstrated by the fact that $G_0\tilde{t}IQ$ is a kernel of the Fredholm type; this follows easily for the pure BCM since $\tilde{t}$ is then separable, while the addition of a reasonable external potential can only improve its square integrability.[7] The separability of $G_0\tilde{t}IQ$ for the pure BCM reduces Eq. (21) to an integral equation in a single vector variable analogous to that arising in the usual formalism from a separable potential, and thus permits an exact solution. This feature is apparently unique for this model since the full BCM t matrix is nonseparable.

In summary, we have demonstrated that the usual Faddeev formalism must and can be restated in the case of singular core interactions. In doing so, we have shown that the problem then reduces to the solution of a new equation, Eq. (21), for the operator M^{ext}. The new formalism is such that it guarantees three-particle unitarity and the correct behavior of the interior wave function, Eq. (20). As a bonus, the nonseparable pure BCM interaction is reduced to an easily soluble three-body problem.

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[1]D. D. Brayshaw, Phys. Rev. C 3, 35 (1971), hereafter referred to as I.

[2]L. D. Faddeev, Zh. Eksp. Teor. Fiz. 39, 1459 (1960) [Sov. Phys. JETP 12, 1014 (1961)].

[3]All of our remarks concerning the BCM will also apply to the hard core, which is merely a special case.

[4]This result is not wholly surprising. One might infer it, for example, from the exterior-interior wave-function formulation of H. P. Noyes, Phys. Rev. Lett. 23, 1201 (1969). In the BCM case the interior wave function must vanish identically. A one-dimensional equation has also been derived in coordinate space for the special case of a pure hard core and s waves only by V. Efimov, Yad. Fiz. 10, 107 (1969) [Sov. J. Nucl. Phys. 10, 62 (1970)].

[5]H. Feshbach and E. Lomon, Phys. Rev. 102, 891 (1956).

[6]For greater detail as to our conventions see D. D. Brayshaw, Phys. Rev. 176, 1855 (1968).

[7]It is necessary, however, to exercise some care in evaluating the product $\tilde{t}^{\,\mathrm{BC}}IQ$; one must keep in mind the limiting procedure used to derive t^{BC} in I.