Cell Model of a Bose-Condensed Solid*

William J. Mullin
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01002
(Received 21 December 1970)

A cell model of a Boson quantum system is described. The model has three phases at $T=0$ K: normal crystal, Bose-condensed liquid, and Bose-condensed crystal. The excitation spectrum for the Bose-condensed crystal has two branches, a gapless vacancy wave mode and a vacancy-interstitial wave mode with a gap. It is proposed that this phase might be detected in ^{4}He near the liquid-solid transition line.

Chester,[1] Andreev and Lifshitz,[2] and Leggett[3] have speculated that a quantum crystal (perhaps ^{4}He) might undergo a Bose-Einstein condensation and exhibit "superfluidity." I wish to present a cell model of a Bose system which has three possible phases: "normal" crystal (NC); Bose-condensed liquid (L); and a third phase, having both crystalline order and a Bose condensation, which we call a "supersolid" (SS).

The basic properties of the NC phase of the model were discussed by the author in a previous paper[4] (hereafter called I). The cell-model Hamiltonian[5] used here is

$$H = H_0 + H_1, \tag{1}$$

where

$$H_0 = t\sum_{\vec{R}_\alpha} n_{\vec{R}_\alpha} + t\sum_{\vec{R}_\beta} n_{\vec{R}_\beta} + \varphi_1 \sum_{\vec{R}_\alpha, \vec{a}_1} n_{\vec{R}_\alpha + \vec{a}_1} n_{\vec{R}_\alpha}, \tag{2}$$

$$H_1 = -\tau\Big(\sum_{\vec{R}_\alpha, \vec{a}_1} b_{\vec{R}_\alpha + \vec{a}_1}^{\dagger} b_{\vec{R}_\alpha} + \sum_{\vec{R}_\beta, \vec{a}_1} b_{\vec{R}_\beta + \vec{a}_1}^{\dagger} b_{\vec{R}_\beta}\Big), \tag{3}$$

with $n_R = b_{\vec{R}}^{\dagger} b_{\vec{R}}$. The $b_{\vec{R}}^{\dagger}$ is an operator which creates a particle in the cell at $\vec{R}$. We have divided our lattice into two interpenetrating sublattices α and β. In the NC phase it is assumed that most particles occupy the α sites $\vec{R}_\alpha$, and very few will be found on the β or "interstitial" sites $\vec{R}_\beta$. The kinetic energy due to localization of a particle on an α or a β cell is t. We assume that particles on near-neighbor α and β sites, which are separated by distance $\vec{a}_1$, will interact through a potential φ_1. The term H_1, which arises from the off-diagonal matrix elements of the kinetic energy,[4,5] allows a particle to tunnel from a cell at $\vec{R}$ to a nearest-neighbor cell. The interstitial β sites are needed in the NC phase to allow the tunneling term H_1 to operate because we do not allow double occupation of any single cell to occur. The double occupancy is avoided by imposing *anticommutation* relations[6] on $b_{\vec{R}}$ and $b_{\vec{R}'}^{\dagger}$ for $\vec{R}$ and $\vec{R}'$ the same, but the usual Bose commutation relations for $\vec{R}$ and $\vec{R}'$ different. The anticommutation relations simulate a hard core.

The mixed commutation relations of the $b_{\vec{R}}$'s are those obeyed by Pauli matrices. We use this fact to transform the Hamiltonian into a pseudospin form.[4,6] Write

$$b_{\vec{R}}^{\dagger} = \sigma_{\vec{R}}^{(+)}, \quad b_{\vec{R}} = \sigma_{\vec{R}}^{(-)} \tag{4}$$

$$n_{\vec{R}} = b_{\vec{R}}^{\dagger} b_{\vec{R}} = \tfrac{1}{2}(1 + \sigma_{\vec{R}}^{(z)}). \tag{5}$$

With this identification, an up "spin" corresponds to an occupied site and a down "spin" to an unoccupied one. In I we took $\varphi_1 > 0$ to represent a repulsion between particles on neighboring α and β sites so that the interstitial β's would be mostly unoccupied. We also included an attractive potential, $-\varphi_2$, between any two nearest α sites. For small enough values of τ (τ is a function of volume) the NC phase is stable and describable as the analog of an anisotropic Heisenberg antiferromagnet in an external field [Fig. 1(a)]. The

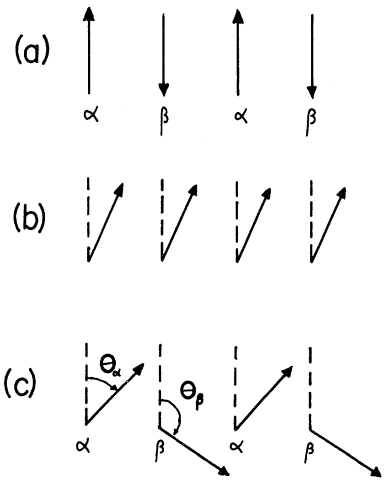

FIG. 1. Spin-analog configurations; an up spin represents an occupied cell, a down-spin an empty one. (a) The NC phase has particles on regular (α) sites and none on interstitials (β). The ground state has zero-point motion and exchange via the interstitials. (b) The superfluid liquid (L) phase. A canted spin implies partial average occupation of a cell and a Bose condensation. (c) The "supersolid" (SS) phase.

ground-state energy includes antiferromagnetic spin fluctuations which can be interpreted as particle zero-point motion from regular to interstitial sites and particle exchange via intermediate interstitial states. The excited states include two branches, each with a gap. One branch represents vacancy-wave states and the other interstitial-wave states.

For sufficiently large τ the NC phase becomes unstable and the "magnetization" tips away from complete alignment along the z axis to an angle having a component $M_\perp$ in the x-y plane [Figs. 1(b) and 1(c)]. A nonzero $M_\perp$ implies a Bose-Einstein condensation,[6] since the order parameter[7] is

$$\langle \psi^+(\vec{r})\rangle \sim \langle b_{\vec{R}}^\dagger\rangle = \langle \sigma_{\vec{R}}^{-(+)}\rangle \sim M_\perp. \tag{6}$$

If $\varphi_1 < 0$, the stable phase is that shown in Fig. 1(b) in which all spins have equal z components. Since the density of particles in a cell at $\vec{R}$ depends on $\sigma_{\vec{R}}^{-(z)}$ [Eq. (5)], this state has uniform density everywhere in the ground state. Thus this corresponds to the liquid (L) phase. If $\varphi_1 > 0$, the z components of neighboring spins would prefer to be antiparallel while the tunneling term aligns the x-y components parallel to one another. Thus the state (SS) shown in Fig. 1(c) is energetically favored. The SS state is crystalline because the density ($\sim\sigma_R^{(z)}$) is not uniform but varies periodically from the α to the β site. Moreover, the system has a Bose condensation since $M_\perp \neq 0$.

The mathematical treatment of H for zero temperature proceeds as follows:

(1) Since the spin-wave approximation that we use for the L or SS phase gives us a Hamiltonian which does not commute with the number operation N, we introduce a chemical potential μ and study $H' = H - \mu N$.

(2) To treat arbitrary α-spin angles θ_α and β-spin angles θ_β we introduce a coordinate transformation to axes ξ_α, η_α, and ζ_α and ξ_β, η_β, and ζ_β:

$$\begin{aligned}
\sigma_i^{(x)} &= \sigma_i^{(\xi)} C_i + \sigma_i^{(\zeta)} S_i, \\
\sigma_i^{(z)} &= -\sigma_i^{(\xi)} S_i + \sigma_i^{(\zeta)} C_i, \\
\sigma_i^{(y)} &= \sigma_i^{(\eta)},
\end{aligned} \tag{7}$$

where $i = \alpha, \beta$ and

$$S_i = \sin\theta_i, \quad C_i = \cos\theta_i. \tag{8}$$

We take an α spin to be quantized along ζ_α and a β along ζ_β. We assume that $M_\perp$ is in the x direction.

(3) After introducing Eqs. (7), the molecular-field or Hartree approximation for $T = 0$ K can be derived by setting $\langle\sigma^{(\zeta)}\rangle = 1$, $\langle\sigma^{(\xi)}\rangle = \langle\sigma^{(\eta)}\rangle = 0$. We find that the Hartree value of H' is

$$E_H' = (t-\mu)N_H + \tfrac{1}{2} z_1 \varphi_1 N_\alpha N_\beta / M - \tfrac{1}{2} z_1 \tau M S_\alpha S_\beta, \tag{9}$$

where z_1 is the number of β sites around an α site, and M is the total number of α (or β) cells. N_H is the Hartree expectation value for the number operator,

$$N_H = N_\alpha + N_\beta, \tag{10}$$

where N_α (N_β) is the Hartree average number of particles in all α (β) cells, i.e., from Eq. (5),

$$\begin{aligned}
N_\alpha &= \tfrac{1}{2} M (1 + C_\alpha), \\
N_\beta &= \tfrac{1}{2} M (1 + C_\beta).
\end{aligned} \tag{11}$$

The terms in E_H' are the kinetic energy and chemical potential, the interaction of particles on α and β cells, and the tunneling energy.

(4) One can find conditions on θ_α, θ_β, and μ by minimizing E_H' with respect to θ_α and θ_β. The two conditions are

$$S_\alpha (t + p - \mu) + p S_\alpha C_\beta + z_1 \tau S_\beta C_\alpha = 0, \tag{12}$$

$$S_\beta (t + p - \mu) + p S_\beta C_\alpha + z_1 \tau S_\alpha C_\beta = 0, \tag{13}$$

where $p = \tfrac{1}{2} z_1 \varphi_1$. One obvious solution to these equation is $S_\alpha = S_\beta = 0$. This corresponds to the NC phase. If we assume that $S_\alpha \neq 0$, $S_\beta \neq 0$, and then eliminate μ, we find that *either*

$$C_\alpha - C_\beta = 0 \tag{14}$$

or

$$z_1 \tau (C_\alpha C_\beta + 1) - p S_\alpha S_\beta = 0. \tag{15}$$

The condition $C_\alpha = C_\beta$ corresponds to the L phase. If $C_\alpha \neq C_\beta$, Eq. (15) holds, which is the condition for the SS phase.

(5) Next we follow the usual spin-wave analysis by introducing the Fourier-transformed spin-deviation operators. We find that the resulting form of H' has linear terms in the spin-deviation operators, which correspond to nonequilibrium positions of the spins. The coefficients of these terms vanish when we impose the conditions Eqs. (12) and (13). There is also a troublesome quadratic term which vanishes for the SS phase by Eq. (15) and leaves us with a Hamiltonian that can be diagonalized by a Bogoliubov transformation. The excitation spectrum of the system has

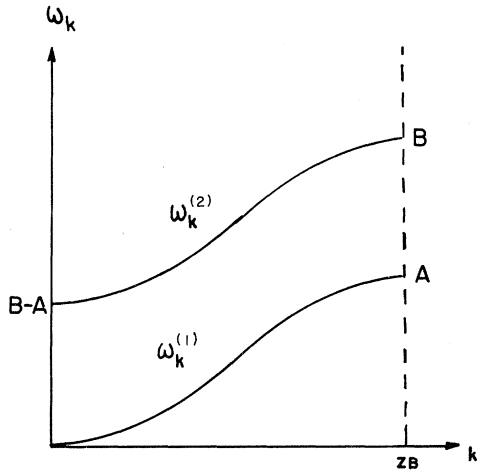

FIG. 2. Qualitative illustration of the excitation spectra of the SS phase: lower mode, vacancy wave; upper mode, vacancy-interstitial pair wave. ZB is the zone boundary. A and B are defined in the text.

two branches given by

$$\omega_k{}^{(1)} = W[1-(\tau\gamma_k/W)^2]^{1/2} + \tfrac{1}{2}(A-B), \tag{16}$$

$$\omega_k{}^{(2)} = W[1-(\tau\gamma_k/W)^2]^{1/2} - \tfrac{1}{2}(A-B); \tag{17}$$

$$A = z_1\tau S_\alpha/S_\beta, \quad B = z_1\tau S_\beta/S_\alpha, \tag{18}$$

$$W = \tfrac{1}{2}(A+B), \quad \gamma_k = \textstyle\sum_{a_1} e^{i\vec{k}\cdot\vec{a}_1}. \tag{19}$$

In deducing the expressions for A and B we have made use of Eqs. (12) and (13). A is the energy required to reverse a pseudospin on an α site; thus if $C_\alpha > 0$, it is the energy necessary to diminish the average occupation number of an α cell by a fraction C_α. Similarly, if $C_\beta < 0$, B is the energy necessary to increase the average occupation of a β cell by C_β.

The lower branch of the energy spectrum (Fig. 2) is a "ferromagnetic" mode, in which neighboring α and β pseudospins swivel in phase, and the upper branch is an "antiferromagnetic" mode in which they swivel 180° out of phase. The small-k behavior of each of these modes is found to be

$$\omega_k{}^{(1)} = z_1\tau\frac{2}{3}\frac{S_\alpha S_\beta}{S_\beta{}^2 - S_\alpha{}^2}\, a_1{}^2k^2, \tag{20}$$

$$\omega_k{}^{(2)} = z_1\tau\left[\frac{S_\beta{}^2 - S_\alpha{}^2}{S_\alpha S_\beta} + \frac{2}{3}\frac{S_\alpha S_\beta}{S_\beta{}^2 - S_\alpha{}^2}\, a_1{}^2k^2\right], \tag{21}$$

where we have assumed $S_\beta{}^2 > S_\alpha{}^2$. Near the zone boundary $\gamma_k \approx 0$, and we have $\omega_k{}^{(1)} = A$ and $\omega_k{}^{(2)} = B$. The antiferromagnetic mode corresponds to the decrease of α-cell densities and the increase of β-cell densities; thus such an excitation corresponds to a vacancy-interstitial pair wave. The ferromagnetic mode for $k \approx 0$ involves a decrease in the density at all sites and corresponds to a gapless vacancy wave. Note that each excitation moves with free single-particle-like behavior, $\omega \sim k^2$.

If $S_\alpha = S_\beta$, we should have the liquid state and the above formulas break down. Indeed if we go back and recompute we find that for $S_\alpha = S_\beta$ both branches coalesce into one which has the appropriate phononlike behavior[6] for small k.

Previous discussions[1,2] of Bose condensed solids have involved ground-state vacancies. The SS phase has less than one particle, on the average, in any cell. We can view this as the presence of vacancies in the ground state; of course, these are nonlocalized vacancies, and no one site is completely empty. Since the vacancy number is macroscopic and the vacancy position probability is uniform over the α (or β) cells, it is proper to consider the SS state as a Bose condensation of vacancies in the $k = 0$ state. This description is similar to that of Andreev and Lifshitz[2] and we believe that our model represents an explicit example of their general considerations.

In addition to possible "superfluidity,"[2] the most obvious feature of the model that might be observed experimentally is the dispersion curve of the "antiferromagnetic" mode which has a gap at $k = 0$. Since the SS phase requires a repulsive interaction between neighboring α and β cells (as does the NC phase), we might expect to find the SS phase, if it exists in nature, close to the liquid-solid transition point. Unfortunately, the model does not specify the number of ground-state vacancies (or, equivalently, the ratio N/M), and so θ_α and θ_β are unknown. However, we might estimate the gap very roughly as $\approx z_1\tau \approx 12$ K by using the $z_1\tau$ value from I for the NC phase.[4]

The recent Letter of Guyer,[8] which discounts the possibility of finding superfluidity in solid ^{4}He with existing low-temperature facilities, is based on an analysis of data taken in the normal-crystal phase. We agree that no superfluidity should be found in that phase. However, we have proposed in this Letter that another phase may be possible which has both crystalline order and a Bose condensation. If this phase exists in ^{4}He, presumably it occupies such a small region of the phase diagram that it has not yet been observed.

*Work supported in part by the U. S. Office of Naval

Research under Contract No. N0001467-A-0230-0003.

[1]G. V. Chester, Phys. Rev. A 2, 256 (1970).

[2]A. F. Andreev and I. M. Lifshitz, Zh. Eksp. Teor. Fiz. 56, 2057 (1969) [Sov. Phys. JETP 29, 1107 (1969)].

[3]A. J. Leggett, Phys. Rev. Lett. 25, 1543 (1970).

[4]W. J. Mullin, to be published.

[5]Similar cell-model Hamiltonians have been used by T. Matsubara and H. Matsuda, Progr. Theor. Phys. 16, 416, 569 (1956), and 17, 19 (1957); R. Whitlock and P. Zilsel, Phys. Rev. 131, 2409 (1963); P. R. Zilsel, Phys. Rev. Lett. 15, 476 (1965); H. A. Gersch and J. M. Tanner, Phys. Rev. 139 A1769 (1965); R. A. Guyer and L. I. Zane, Phys. Rev. Lett. 24, 660 (1970), and to be published; R. Balakrishnan and R. V. Lange, Phys. Rev. A 3, 496 (1971).

[6]See the first two sets of papers of Ref. 5. Also see M. E. Fisher, Rep. Progr. Phys. 30, 615 (1967), Sect. 4.

[7]P. W. Anderson, Rev. Mod. Phys. 38, 298 (1966).

[8]R. A. Guyer, Phys. Rev. Lett. 26, 174 (1971).

Anomalous Skin Effect of Microwaves Incident on Magnetoplasmas

Kazuo Minami and Susumu Takeda
Department of Electrical Engineering, Nagoya University, Nagoya, Japan
(Received 9 February 1971)

The anomalous skin effect of microwaves is experimentally investigated for a high-density gaseous plasma near the electron cyclotron resonance and is compared with a theoretical prediction.

The increase of the penetration depth of the electromagnetic field in a gaseous plasma due to the thermal electron motion is called the anomalous skin effect[1,2] just as in metals.[3] Shafranov[4] analyzed the penetration of a wave into a collision-free, magnetized plasma uniformly distributed in a half-space. Platzman and Buchsbaum[5] extended the analysis, showing some numerical results. Weibel[2] also strictly analyzed the anomalous skin effect in a uniform, nonmagnetized plasma, referring to a previous experiment.[6] Drummond[7] extended Weibel's theory to a plasma with a magnetic field. The experimental works[1,6,8] were limited to plasmas where the collision frequency ν was higher than the applied frequency ω.

The classical penetration depth of the electromagnetic field into high-density plasmas in a cutoff region d defined by $1/\mathrm{Im}(k)$ is derived from the dielectric constant of the plasma, $1-(\omega_p/\omega)^2 \times(1+i\nu/\omega)^{-1}$, as

$$d \sim (2\nu/\omega)^{1/2} c/\omega_p \text{ for } \omega \ll \nu \quad (1)$$

and

$$d \sim c/\omega_p \text{ for } \omega \gg \nu, \quad (2)$$

where the thermal motion of electrons is neglected for plasmas and the plasma frequency ω_p is much higher than ω. This case is called the cold-plasma theory. Equation (1) is the same as the well-known classical skin depth in metals. For Eq. (2), the plasma behaves like a dielectric material with respect to the electromagnetic field.

When d is smaller than v_{th}/ν for $\omega \ll \nu$, the thermal velocity of electrons v_{th} should be taken into account for the calculation of the skin depth. For $\omega \gg \nu$, v_{th}/ν should be replaced by v_{th}/ω. These results are expected from the theoretical analysis.[2,7] The effect of a magnetic field is quite different for the case $\omega \gg \nu$ from that for the case $\omega \ll \nu$. In the latter, the magnetic field is not too sensitive to the skin depth because the plasma is collision dominated. On the other hand, for $\omega \gg \nu$, the characteristic of the plasma medium is generally expected to be strongly affected by the magnetic field, particularly near the electron cyclotron resonance.

As the gas pressure in the positive column, where $\omega \ll \nu$, decreases, the skin depth is found[1,8] to become larger than the classical value because of the increased v_{th}/ν. However, no experiments have been reported on a plasma with a magnetic field for $\omega \gg \nu$. Thus, the detection of the anomalous skin effect for $\omega \gg \nu$ by using a microwave reflection technique[9] is tried here.

A right-hand circularly polarized wave (R wave) is used, since the anomalous skin effect is expected[7] to be dominant near the resonance even for relatively low electron temperatures. When the R wave is normally incident on a plasma boundary with perpendicular magnetic field, the wave reflection coefficient is calculated from the electric field expressed in Eq. (14) of Ref. 2. From the coefficient, we obtain a refractive in-