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[1]P. Sonderegger, J. Kirz, O. Guisan, P. Falk-Vairant, C. Bruneton, P. Borgeaud, A. V. Stirling, C. Caverzasio, J. P. Guillaud, M. Yvert, and B. Amblard, Phys. Lett. 20, 75 (1966).

[2]R. C. Chase, E. Coleman, H. W. J. Courant, E. Marquit, E. W. Petraske, H. F. Romer, and K. Ruddick, Phys. Rev. D 2, 2588 (1970)

[3]H. R. Crouch *et al.*, Phys. Rev. Lett. 21, 849 (1968); M. A. Wahlig and I. Manelli, Phys. Rev. 168, 1515 (1968).

[4]V. Barger and R. J. N. Phillips, Phys. Rev. Lett. 22, 116 (1969).

[5]D. R. Rust, P. N. Kirk, R. A. Lundy, C. E. W. Ward, D. D. Yovanovitch, S. M. Pruss, C. W. Akerlof, K. S. Han, D. I. Meyer, and P. Schmueser, Phys. Rev. Lett. 24, 1361 (1970); B. B. Brabson, R. R. Crittenden, R. M. Heinz, R. C. Kammerud, H. A. Neal, H. W. Paik, and R. A. Sidwell, Phys. Rev. Lett. 25, 553 (1970).

High-Energy Bounds on Scattering Amplitudes and Oscillations in the Diffraction-Peak Region

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Rigorous bounds are established on the absorptive part of the scattering amplitude, $A(s,t)$, for t real and within the Lehman-Martin ellipse. This result is used to prove an upper bound on $[d \ln A(s,t)/dt]_{t=0}$, and to show that $A(s,t)$ cannot have a zero in the region $0 > t > -4t_0/(\ln s)^2$ for $s\to\infty$ ($\sqrt{t_0} = 2m_\pi$ for $\pi\pi$, πN, KN, and NN scattering). No assumption is made about the high-energy behavior of the total cross sections.

The Serpukhov data on particle and antiparticle total cross sections[1] $\sigma_{\rm tot}(s)$ has pointed to a possible failure of the Pomeranchuk theorem.[2] This has led to a reanalysis of the "proofs" and to the derivation of new Pomeranchuk-like theorems.[3] In trying to understand this possible failure in terms of a Regge picture, Finkelstein produced an interesting model involving Regge cuts.[4] This model amplitude turns out to have infinite number of oscillations in the physical region for the scattering. It has been suggested that this feature may be general, provided the particle and antiparticle total cross sections tend to different finite limits.[5]

The purpose of the present investigation is to derive restrictions on high-energy behavior of the scattering amplitudes and, in particular, on the location of the zeros in the physical region, using only general considerations. We make use of only (i) unitarity, (ii) analyticity within the Lehman-Martin ellipse, and (iii) the Jin-Martin upper bound

$$A(s,t_0) \underset{s\to\infty}{<} S^2, \tag{1.1}$$

where $A(s,t)$ is the absorptive part of the scattering amplitude and s and t are, respectively, the squared c.m. energy and momentum transfer variables.[6] The major axis of the Lehman-Martin ellipse is given by $2(1+t_0/2k^2)$, where k is the c.m. momentum. We have $\sqrt{t_0}=2m_\pi$ for $\pi\pi$, πN, πK, KN, and NN scattering. *We shall not make any assumptions about the high-energy behavior of the total cross sections.*

The basic theorem. – The absorptive part of the scattering amplitude, $A(s,t)$, has the partial-wave expansion (for $t_0 > t$)

$$A(s,t) = \frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}\,a_l P_l(z), \tag{2.1}$$

where $t=-2k^2(1-z)$ and $\mathrm{Im}\,a_l \geq 0$ from unitarity. All the results discussed in this paper are derived from the following basic result:

Theorem 1.

$$\max\{J_{M,N}{}^L(t,t_0)|M>L\geq N\} \geq \frac{A(s,t)}{A(s,t_0)} \geq \min\{J_{M,N}{}^L(t,t_0)|M>L\geq N\} \quad (M,N=0,1,2,\cdots), \tag{2.2}$$

where

$$[P_M(x)-P_N(x)]J_{M,N}{}^L(t,t_0)=P_N(z)[P_M(x)-(1-\epsilon)P_L(x)-\epsilon P_{L+1}(x)]+P_M(z)[(1-\epsilon)P_L(x)+\epsilon P_{L+1}(x)-P_N(x)],$$

$$t_0=-2k^2(1-x), \tag{2.3}$$

and the non-negative integer L and the number ϵ $(1>\epsilon\geqslant 0)$ are determined by

$$A(s,t_0)=A(s,0)[(1-\epsilon)P_L(x)+\epsilon P_{L+1}(x)]. \tag{2.4}$$

Proof: Consider the problem of finding the extremum of $A(s,t)$ given by (2.1) subject to the constraints

$$\left.\begin{aligned} &A(s,t_0)=(s^{1/2}/k)\sum_{l=0}^{\infty}(2l+1)\mathrm{Im}a_l(s)P_l(x),\\ &A(s,0)=(s^{1/2}/k)\sum_{l=0}(2l+1)\mathrm{Im}a_l(s),\\ &\mathrm{Im}a_l(s)\geqslant 0 \end{aligned}\right\} \tag{2.5}$$

with $A(s,t_0)$ and $A(s,0)$ regarded as given quantities. The set C of the solutions of the constraint equations (2.5) form a convex set. An extreme point of this set C is conveniently labeled by specifying two integers. The extreme point $S_{M,N}$ (M,N integral, $M>N$) of the set C is given by solving (2.5) with the proviso that $\mathrm{Im}a_l=0$ for $l\neq M,N$, i.e.,

$$\left.\begin{aligned} &\mathrm{Im}a_N=\frac{kA(s,0)}{(2N+1)\sqrt{s}}\,\frac{P_M(x)-(1-\epsilon)P_L(x)-\epsilon P_{L+1}(x)}{P_M(x)-P_N(x)},\\ &\mathrm{Im}a_M=\frac{kA(s,0)}{(2M+1)\sqrt{s}}\,\frac{(1-\epsilon)P_L(x)+\epsilon P_{L+1}(x)-P_N(x)}{P_M(x)-P_N(x)},\\ &\mathrm{Im}a_l=0 \text{ for } l\neq M,N, \end{aligned}\right\} \tag{2.6}$$

where

$$M>L\geqslant N.$$

The range of M and N is fixed by positivity of $\mathrm{Im}a_l$. The value of $A(s,t)/A(s,0)$ at the point $S_{M,N}$ of the set C is given by $J_{M,N}{}^L(t,t_0)$. Since the set C is convex the extrema of the linear form $A(s,t)$ will be attained at one of the extreme points.[7] Hence the theorem follows.

We also note here the inequality

$$L \underset{s\to\infty}{>} (s/4t_0)^{1/2}\ln(s/\sigma_{\mathrm{tot}}) \tag{2.7}$$

which follows on using the Jin-Martin bound (1.1) and the relation (2.4). Theorem 1 has interesting consequences only for an upper bound on $A(s,t)$ in the region $t_0>t>0$ and for a lower bound on $A(s,t)$ for $0>t$. We shall discuss only these and the consequences derivable from them.

An upper bound on $A(s,t)$ in the region $t_0>t>0$, i.e., $x>z>1$.—Theorem 2(A).

$$J_{L+1,L}{}^L(t,t_0)\equiv(1-\epsilon)P_L(z)+\epsilon P_{L+1}(z)\geqslant A(s,t)/A(s,0) \text{ for } t_0\geqslant t\geqslant 0. \tag{3.1}$$

Proof: Define a function $f(\xi)$ by the following rule:

$$f(\xi)=-[(1-\epsilon)P_n(z)+\epsilon P_{n+1}(z)] \tag{3.2}$$

for

$$\xi=(1-\epsilon)P_n(x)+\epsilon P_{n+1}(x) \text{ and } x>z>1;\ n=0,1,2,\cdots.$$

The function $f(\xi)$ is a continuous function of ξ for $\xi>1$. We have

$$f'(\xi)=-\frac{P_{n+1}(z)-P_n(z)}{P_{n+1}(x)-P_n(x)}=-(z-1)\sum_{k=0}^{n}(2k+1)P_k(z)\Big[(x-1)\sum_{k=0}^{n}(2k+1)P_k(x)\Big]^{-1} \text{ for } P_{n+1}(x)>\xi>P_n(x). \tag{3.3}$$

Therefore

$$\lim_{\lambda\to 0^+}[f'(\xi+\lambda)-f'(\xi-\lambda)] = \frac{(z-1)(2r+1)\sum_{r=0}^{n-1}(2r+1)[P_r(z)P_n(x)-P_r(x)P_n(z)]}{(x-1)\sum_{k=0}^{n}(2k+1)P_k(x)\sum_{q=0}^{n-1}(2k+1)P_q(x)} \quad \text{for } \xi = P_n(x),$$

$$= 0 \text{ for } \xi \neq P_n(x).$$

Using $P_r(z)P_n(x) \geq P_r(x)P_n(z)$ for $x \geq z \geq 1$ and $n \geq r$, we obtain

$$\lim_{\lambda\to 0^+}[f'(\xi+\lambda)-f'(\xi-\lambda)] \geq 0 \text{ for } \xi > 1. \tag{3.4}$$

We thus see that the function $f(\xi)$ is a convex function of ξ. Therefore

$$(y_2-y)f(y_1)+(y-y_1)f(y_2) \geq (y_2-y_1)f(y) \text{ for all } y_2 \geq y \geq y_1 > 1. \tag{3.5}$$

Choosing $y_2 = P_M(x)$, $y = (1-\epsilon)P_L(x)+\epsilon P_{L+1}(x)$, $y_1 = P_N(x)$, we obtain

$$J_{L+1,L}{}^L(t,t_0) \underset{(t_0 \geq t \geq 0)}{\geq} J_{M,N}{}^L(t,t_0) \text{ for } M > L \geq N. \tag{3.6}$$

Then Theorem 2(A) follows from Theorem 1 on using (3.6).

Using Theorem 2(A) and the result (2.7) we obtain *Theorem 2(B)*:

$$A(s,0)I_0[(t/t_0)^{1/2}\ln(s/\sigma_{\text{tot}})] \underset{s\to\infty,\, t_0 > t > 0}{\geq} A(s,t), \tag{3.7}$$

where the function $I_0(\xi)$ is a modified Bessel function.

The upper bound given by Theorem 2 improves a result of Martin.[8] It also has the important consequence

$$1+(t/t_0)^{1/2} \geq \alpha(t) \text{ for } t_0 \geq t \geq 0, \tag{3.8}$$

where $\alpha(t)$ is the position of the J-plane singularity giving the high-s behavior of $A(s,t)$. This result was first noticed by Martin.[8] It has, more recently, also been rediscovered by Arafune and Sugawara who also obtained Theorem 2(B) with extra assumptions of (i) constancy and (ii) inequality of the high-energy particle and antiparticle total cross sections.[9] These assumptions are not needed.

A lower bound on the "diffraction peak width."—We now use Theorems 2(A) and 2(B) to derive a lower bound on the "diffraction peak width" w, defined by

$$w^{-1} = [d\ln A(s,t)/dt]_{t=0}. \tag{4.1}$$

We have from Theorem 2(A) the result

$$\{(1-\epsilon)[P_L(z)-1]+\epsilon[P_{L+1}(z)-1]\}/t \underset{t_0 \geq t \geq 0}{\geq} \{A(s,t)-A(s,0)\}/tA(s,0).$$

Taking the limit $t\to 0^+$, we obtain

$$(L+2\epsilon)(L+1) \geq (4k^2)w^{-1}. \tag{4.2}$$

Combining the result (4.2) and (2.7) we obtain *Theorem 3*:

$$(1/4t_0)[\ln(s/\sigma_{\text{tot}})]^2 \underset{s\to\infty}{\gtrsim} w^{-1}. \tag{4.3}$$

Theorem (3) removes the major arbitrariness in the upper bound on w^{-1} given by Kinoshita,[10] i.e., constant $\times (\ln s)^2 \underset{s\to\infty}{\gtrsim} w^{-1}$.

A lower bound on $A(s,t)$ in the physical region, i.e., $0 \geq t$.—The function $P_l(z)$ is an infinitely oscillating function of l in the physical region $(1 \geq z \geq -1)$. It is therefore not possible to evaluate explicitly the lower bound given by Theorem 1 for all physical values of z. A number of useful properties of the lower bound can, however, be established and we proceed to discuss them now.

(i) *Theorem 4(A).* There is a region $0 \geq t > t_m$ in which

$$A(s,t)/A(s,0) \geq (1-\epsilon)P_L(z) + \epsilon P_{L+1}(z) \equiv J_{L+1,L}{}^L(t,t_0). \tag{5.1}$$

Proof: We have the inequality (3.6) for $t > 0$. Further, the two sides of the inequality are equal at $t = 0$. Since both sides of this inequality are polynomials in z it follows that

$$J_{M,N}{}^L(t,t_0) \geq J_{L+1,L}{}^L(t,t_0)$$

for $M > L \geq N$ and $0 \geq t > t_{M,N}$, where $t_{M,N}$ is the largest negative value of t for which the two sides are equal for M, N such that $M > L \geq N$ and $(M,N) \neq (L+1, L)$. Further, $t_{M,N} < 0$ for all relevant M and N. Setting

$$t_m = \max\{t_{M,N} \mid M > L \geq N; (M,N) \neq (L+1,L)\}, \tag{5.2}$$

the theorem follows with $t_m \neq 0$.

(ii) *Theorem 4(B).* The lower bound on $A(s,t)$ for $0 > t$ given by Theorem 1 is negative definite for $L \geq l_0$, where l_0 is the lowest positive integral value of l for which $P_l(z) \leq 0$.

Proof: Choose a value of $M > L$ such that $P_M(z) < 0$ and let $N = l_0$. This can always be done in view of infinite oscillations in l of $P_l(z)$ for $1 > z > -1$. The function $J_{M,N}{}^L(t,t_0)$ is negative definite for this choice of M and N. Hence the assertion.

A corollary of this result is that t_m is finite.

(iii) In order to prove the next result we need the following presumably new inequality:

$$g_l(z) \equiv P_l(z) - [1 - \tfrac{1}{2}l(l+1)(1-z)] \geq 0 \text{ for } 1 \geq z \geq -1. \tag{5.3}$$

Consider

$$g_{l+1}(z) - g_l(z) = P_{l+1}(z) - P_l(z) + (l+1)(1-z) = (1-z)\sum_{k=0}^{l}(2k+1)[1-P_k(z)]/(l+1).$$

Since $1 \geq P_l(z)$ for $1 > z > -1$, it follows that $g_l(z)$ is a nondecreasing function of l. Noting that $g_0(z) = 0$ the result (5.3) follows. Using the inequality (5.3) in the lower bound given by Theorem 1 we get for $1 \geq z \geq -1$ the bound

$$A(s,t) \geq A(s,0)\left[1 - \max\{K_{M,N}{}^L(t_0) \mid M > L \geq N\}\left(\frac{1-z}{2}\right)\right],$$

where

$$[P_M(x) - P_N(x)]K_{M,N}{}^L(t_0) \equiv N(N+1)[P_M(x) - (1-\epsilon)P_L(x) - \epsilon P_{L+1}(x)]$$
$$+ M(M+1)[(1-\epsilon)P_L(x) - \epsilon P_{L+1}(x) - P_N(x)].$$

We can, again using the method used in proving Theorem 2(A), show that

$$(L+1)(L+2\epsilon) \geq K_{M,N}{}^L(t_0) \text{ for } M > L \geq N.$$

We therefore have *Theorem 4(C):*

$$A(s,t) \geq A(s,0)[-1(L+2\epsilon)(L+1)(1-z)/2] \tag{5.4}$$

and

$$A(s,t) \underset{s\to\infty}{\geq} A(s,0)\left\{1 + \left(\frac{t}{4t_0}\right)\left[\ln\left(\frac{s}{\sigma_{\mathrm{tot}}}\right)\right]^2\right\} \text{ for } 1 \geq z \geq -1. \tag{5.5}$$

Oscillations.—Combining Theorems 4(B) and 4(C) we get the result that the lower bound on $A(s,t)$ given by Theorem 1 is positive in a region

$$0 \geq t > \tau,$$

where

$$(-4t_0) \geq \tau[\ln(s/\sigma_{\mathrm{tot}})]^2 \geq -j_1{}^2 t$$

for $s \to \infty$, where j_1 is the position of the first zero of $J_0(\xi) = 0$, i.e., $j_1 = 2.405\cdots$. Thus the amplitude $A(s, t)$ cannot have oscillations in this region.

[1]J. V. Allaby *et al.*, Phys. Lett. 30B, 500 (1969).

[2]I. Ya. Pomeranchuk, Zh. Eksp. Teor. Fiz. 34, 725 (1958) [Sov. Phys. JETP 7, 99 (1958)].

[3]S. M. Roy and V. Singh, Phys. Lett. 32B, 50 (1970).

[4]J. Finkelstein, Phys. Rev. Lett. 24, 172 (1970).

[5]See, e.g., R. C. Casella, Phys. Rev. Lett. 24, 1463 (1970).

[6]For a recent review of this subject, see, e.g., V. Singh, "Exact results in the analytic *S*-matrix theory of strong interactions for high energies," in Proceedings of the Symposium on "The Past Decade in Particle Theory" (to be published).

[7]S. I. Gass, *Linear Programming* (McGraw-Hill, New York, 1958), Chap. 3.

[8]A. Martin, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorhouse (Oliver and Boyd, Edinburgh, Scotland, 1963).

[9]J. Arafune and H. Sugaware, Phys. Rev. Lett. 25, 1516 (1970).

[10]T. Kinoshita, in *Lectures in Theoretical Physics*, edited by W. E. Brittin, A. O. Barut, and M. Guenin (Gordon and Breach, New York, 1966), Vol. IX B.

ERRATUM

EXPERIMENTAL DETERMINATION OF THE PAIR SUSCEPTIBILITY OF A SUPERCONDUCTOR. J. T. Anderson and A. M. Goldman [Phys. Rev. Lett. 25, 743 (1970)].

As a result of a computational error, the dashed line of Fig. 2(c) misrepresents the work of Ref. 2. The accompanying figure is correct and indicates that the zero-bias conductivity based on the work of Ref. 1 is quantitatively similar to that of Ref. 2.

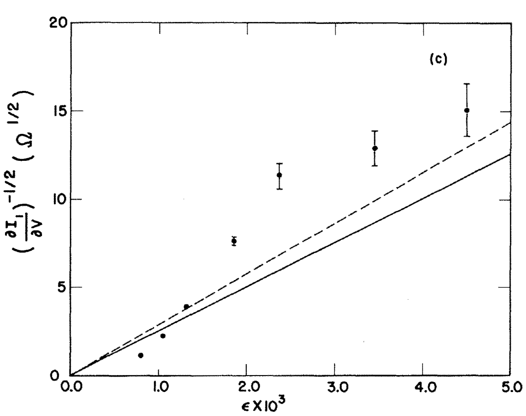

FIG. 2. (c) (revised) $(dI_1/dV)^{-1/2}$ vs ϵ evaluated at zero bias. The solid curve is computed from Eq. (1). The dashed curve is estimated using Eq. (1) of Ref. 10 and is based on the work of Ref. 2.