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"Wave-Mechanical" Model for the Nonadiabatic Loss of Particles from Magnetic-Mirror Traps*

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A model is presented which describes the nonadiabatic loss of particles from magnetic mirror traps through a Schrödinger-like equation where the role of $\hbar$ is played by the "first action invariant." The process is thus found to be analogous to the "tunneling effect" in quantum mechanics. The predictions of the model for the lifetimes compare very well with the experimental results.

Recent experiments[1-3] have determined the average lifetimes of adiabatically trapped charged particles of a given energy and magnetic moment in mirror traps. It is well known that the trapping takes place because of the adiabatic invariance of the magnetic moment of a particle in a slowly varying magnetic field. The magnetic moment $\frac{1}{2}mC_\perp^2/B$ is, apart from a constant, simply the action, $\mu=\frac{1}{2}mC_\perp^2/\Omega$, associated with the cyclotron motion perpendicular to the lines of force. (Here $C_\perp$ is the velocity perpendicular to the magnetic field, and $\Omega=eB/mc$ is the cyclotron frequency.) The escape of particles from the mirror traps is then correctly regarded as being due to departure from adiabaticity. A consequence of this departure is the change in the value of the action μ as the particle traverses a certain region of the magnetic field variation. Expressions for such a change have been obtained in the literature[4,5] but such expressions have not been helpful in the determination of average lifetimes of particles in magnetic mirror traps.

A suggestion usually made[1,2,6] for the mechanism of nonadiabatic escape of particles from magnetic traps is that they suffer a cumulative change in their action invariant until they finally fall into the loss cone and escape. The argument essentially regards the process as a kind of random walk in the μ space and into the loss cone. It must be remembered, however, that even though the process may have the appearance of a random walk, the changes are determined by exact equations of motion, and are therefore far from Markovian.

We wish to emphasize therefore that the nonadiabatic escape is a consequence of the exact equations of motion, and the dynamics contained in these equations constitutes the only legitimate mechanism of escape. To describe the nonadiabatic loss of particles we discard the concept of the adiabatic loss cone for this purpose. Instead, we work in terms of the properties of the exact trajectories. In particular, we introduce the adiabatic action (the time integral of the adiabatic Lagrangian) as a variable to describe the distribution of particles. Even though the exact trajectories are complicated, the variation in action (which is a function of the entire history of the exact orbit) from one trajectory to another and hence the distribution in action have some simple and important properties which we exploit in the formulation of our theory.

Since we wish to describe the experimental situation as closely as possible, we consider a number of particles, with a specified energy E and action invariant μ, equally distributed over the Larmor phase φ initially. (The last condition is inevitably obtained since the injection time is usually much larger than the Larmor period.) Let the exact trajectories of N particles be given by $x_{\parallel i}=x_{\parallel i}(t)$, $\varphi_i=\varphi_i(t)$, and $\mu_i=\mu_i(t)$, $i=1,2,\cdots,N$. Since the nonadiabatic loss takes place along the field lines, we specify only the parallel coordinates $x_{\parallel i}$ of the exact-trajectory end points (the suffix "parallel" will hereafter be dropped). The slight nonadiabaticity which we shall consider guarantees that the distribution in μ at any subsequent time will be centered around the initial value $\bar{\mu}$. We shall, for simplicity, assume our trap to be axisymmetric, and particles to be injected off the axis of symmetry.

We shall, in what follows, use the principle of least action as a guiding principle for the adiabatic motion. We make the observation that the guiding-center equation of motion,

$$mdC_{\parallel}/dt=-\nabla_{\parallel}(\bar{\mu}\Omega), \tag{1}$$

can be obtained by minimizing the action S given by

$$S=\int dt\,(\tfrac{1}{2}mC_{\parallel}^2-\bar{\mu}\Omega)=\int Ldt, \tag{2}$$

$L=(\frac{1}{2}mC_{\parallel}^2-\bar{\mu}\Omega)$ being the Lagrangian. Note that the exact trajectories $x_i=x_i(t)$ are different from and, in the slightly nonadiabatic case, in the neighborhood of the adiabatic one. It then follows that the value of the action S at a given time t as calculated for an exact trajectory will in general be different from the adiabatic value of the action S_A at that space-time point (x,t), but will approach the latter as $\bar{\mu}\to 0$. (The smallness of $\bar{\mu}$ should be formally understood as coming from $\Omega\to\infty$, rather than from the pitch angle $\delta\to 0$, since the latter alters the value of the potential $\bar{\mu}\Omega$.) The same would be true of the infinitesimal changes in action S. Again $\Delta S\to\Delta S_A$, as $\bar{\mu}\to 0$. Furthermore, we wish to emphasize that the difference in the action S from the adiabatic value $S_A(x,t)$, and from one trajectory to another, will be small regardless of their complicated details. In other words, the distribution of particles in the action S would be peaked around the adiabatic value S_A at an adiabatically accessible point (x,t). It may thus be noted that the action appears as a very significant label because for a trajectory to end up outside the trap at the end of a time t, it must have its action different from the minimum for that time. We shall, therefore, find it useful to introduce the action S as a variable.

The problem of determining the probability of nonadiabatic escape of particles then reduces to the problem of determining, at each instant of time, what fraction of the actual trajectories labeled by their actions find their end points outside the adiabatic trap. Instead of the action S, however, we shall introduce a quantity Φ to be called the action phase, differing from it only by the constant factor $\bar{\mu}$ (the value at injection), $\Phi\equiv S/\bar{\mu}=\varphi+\int\frac{1}{2}mC^2dt/\bar{\mu}$. We shall next introduce a function $f(x,\Phi_t,t)$, defined at every point (x,t) to give the (smoothed out) density of trajectory endpoints at the time t per unit interval Δx at x, and with their action phases in $\Delta\Phi_t$ at Φ_t. Clearly then, the particle density at (x,t) is given by

$$G(x,t)=\int d\Phi_t\,f(x,\Phi_t,t); \tag{3}$$

Φ_t denotes a value of the action phase at time t. We can write the following equation for f:

$$f(x,\Phi_{t+\tau},t+\tau)=\int d(\Delta x)f(x-\Delta x,\Phi_t,t)P(x,t+\tau,\Phi_{t+\tau}|x-\Delta x,\Phi_t,t), \tag{4}$$

where P represents the probability that a particle at $(x-\Delta x,t)$ and with action-phase Φ_t goes to the point $(x,t+\tau)$ with action $\Phi_{t+\tau}$. For infinitesimal changes, we simply have $\Phi_{t+\tau}=\Phi_t+L\tau/\bar{\mu}$.

In writing down an expression for the probability function P we first of all note that in the adiabatic limit $\bar{\mu}\to 0$, it must simply be $\delta(\Delta x-(\tau/m)\partial S_A/\partial x)$, so that on integration with respect to Δx and Φ_t, (4) yields the equation of continuity in this limit. A very useful, and natural, choice for P with this property emerges if we write for f a positive definite expression,

$$f(x,\Phi_t,t)=\psi^*(x,\Phi_t,t)\psi(x,\Phi_t,t). \tag{5}$$

Since f as a function of Φ_t is peaked around the adiabatic value $\Phi_A(x,t)$, we can write in the adiabatically accessible region

$$\psi=\textstyle\sum_n\tilde{\psi}(x,n,t)\,\exp\{in[S_t-S_A(x,t)]/\bar{\mu}\}. \tag{6}$$

Consider now the quantity $\psi^*(x,\Phi_{t+\tau}t+\tau)\psi(x-\Delta x,\Phi_t,t)$. Using Eq. (6) and the relation $S_{t+\tau}=S_t+L\tau$, we then have

$$\psi^*(x,\Phi_{t+\tau},t+\tau)\psi(x-\Delta x,\Phi_t,t)=\sum_{n,n'}\tilde{\psi}^*(x,n,t+\tau)\tilde{\psi}(x-\Delta x,n',t)\exp\{-i(n-n')[S_{t+\tau}-S_A(x,t+\tau)]/\bar{\mu}\}\times\exp[-in'(L\tau-\Delta S_A)/\bar{\mu}], \quad (7)$$

where $\Delta S_A=S_A(x,t+\tau)-S_A(x-\Delta x,t)$, and $L=\frac{1}{2}m(\Delta x/\tau)^2-\bar{\mu}\Omega$. An integration with respect to $\Phi_{t+\tau}$ which will eventually be performed results in $n=n'$ in the expression (7). The surviving exponential, on the other hand, amounts to the δ function $\delta(\Delta x-(\tau/m)\partial S_A/\partial x)$ in the limit $\bar{\mu}\to 0$, when an integral involving it is evaluated using the steepest descent method, provided an appropriate normalization factor $A(n)$ is introduced. [This is given by Eq. (9).] We thus see that with suitable normalization the real part of the expression in (7) would serve as a natural choice for P. Note, however, that for $\tau,\Delta x\to 0$ this expression reduces to $|\psi(x,\Phi_t,t)|^2$, which to lowest order can be taken to be $|\psi(x-\Delta x,\Phi_t,t)|^2$. Thus for (7) to be an appropriate expression for P, it must be divided by the latter quantity. Using the expression for P so obtained, in Eq. (4) we get

$$|\psi(x,\Phi_{t+\tau},t+\tau)|^2=\mathrm{Re}\int d(\Delta x)\psi^*(x,\Phi_{t+\tau},t+\tau)\sum_n A(n)\tilde{\psi}(x-\Delta x,n,t)\exp\{in[S_{t+\tau}-L\tau-S_A(x,t)]/\bar{\mu}\}, \quad (8)$$

where Re stands for the real part of the expression to its right. As indicated earlier, it can be easily shown that on integrating Eq. (8) over $\Phi_{t+\tau}$ we get the adiabatic equation of continuity in the limit $\bar{\mu}\to 0$, provided

$$[A(n)]^{-1}=\int d(\Delta x)\exp[-in\tfrac{1}{2}m(\Delta x)^2/\tau\bar{\mu}]=[2\pi\bar{\mu}\tau/(-inm)]^{1/2}. \quad (9)$$

Eq. (8), in the form expressed, is thus an appropriate equation for the adiabatically accessible region. For an arbitrary space point we carry out a Fourier analysis of ψ with respect to Φ_t according to

$$\psi(x,\Phi_t,t)=\sum_n\Psi(x,n,t)\exp(inS_t/\bar{\mu}) \quad (10)$$

rather than according to Eq. (6). Noting that the left-hand side of Eq. (8) is real, we can drop the prefix Re. This immediately gives

$$\psi(x,\Phi_{t+\tau},t+\tau)=\int d(\Delta x)\sum_n A(n)\Psi(x-\Delta x,n,t)\exp[in(S_{t+\tau}-L\tau)/\bar{\mu}]. \quad (11)$$

Fourier analysis of (11) with respect to $\Phi_{t+\tau}$ gives

$$\Psi(x,n,t+\tau)=\int d(\Delta x)A(n)\exp\{-in[\tfrac{1}{2}m(\Delta x)^2/\tau-\bar{\mu}\Omega\tau]/\bar{\mu}\}\Psi(x-\Delta x,n,t). \quad (12)$$

If we now expand both sides about the point (x,t) and use Eq. (9) for $A(n)$, we find the differential equation for $\Psi(x,n,t)$,

$$-i\frac{\bar{\mu}}{n}\frac{\partial\Psi}{\partial t}=-\left(\frac{\bar{\mu}}{n}\right)^2\frac{1}{2m}\frac{\partial^2\Psi}{\partial x^2}+(\bar{\mu}\Omega)\Psi(x,n,t). \quad (13)$$

From the definition (3) of the density (or probability density, depending on the manner of normalization) and from Eq. (10) we obtain the connection between the density and the solutions $\Psi(x,n,t)$ of Eq. (13), namely,

$$G(x,t)=\int d\Phi_t f(x,\Phi_t,t)=\sum_{n\neq 0}\Psi^*(x,n,t)\Psi(x,n,t). \quad (14)$$

The term $n=0$ is to be excluded because from Eq. (12) the only possible solution for $n=0$ is $\Psi(x,n=0,t)=0$.

On comparing Eqs. (6) and (10) we note that $\Psi(x,n,t)\sim\tilde{\psi}\exp(-inS_A/\bar{\mu})$, where S_A is essentially the Hamilton's principal function for the adiabatic motion. It follows that we should seek solutions for Ψ of the form $\Psi\sim\exp(inEt/\bar{\mu})$, where E is the energy of the particle. We see that Eq. (13), with the connection (14), and the form of the solution for Ψ form a close analog of the Schrödinger theory in quantum mechanics. Here the role of $\hbar$ is played by the first action invariant $\bar{\mu}$ (the value at injection). Note also that $\bar{\mu}\Omega$ occuring in the place of the potential is precisely the potential which describes the adiabatic motion. The nonadiabatic escape of particles from the adiabatic traps, which can be calculated from these equations, then appears to be in the nature of the "tunneling effect" in quantum mechanics. It may be pointed out that our derivation is in fact analogous to the Feynman path-integral

Table I. A comparison of the theoretically predicted value $h_{\rm theor}$ of the exponent for $n=1$ [see text, Eq. (15)] with its experimental value $h_{\rm expt}$ for different experiments.

Authors	α^{-1} (cm)	E (keV)	$h_{\rm theor}$	$h_{\rm expt}$
Dubinina *et al.*	25	23.5	$0.17B$	$0.15B$
Ponomarenko *et al.*				
Configuration I	18	9	$0.071B_{\rm max}$	$0.096B_{\rm max}$
Configuration II	14	9	$0.055B_{\rm max}$	$0.051B_{\rm max}$
Configuration III	12	9	$0.047B_{\rm max}$	$0.036B_{\rm max}$

formulation of quantum mechanics.[7]

If we assume the potential $\bar{\mu}\Omega$ to be of the form $\bar{\mu}\Omega=\bar{\mu}\Omega_{\rm max}[\cosh(\alpha x)]^{-2}$ in the region of the mirrors along a certain field line, then the probability of transmission per unit time across the potential hill is given by

$$P=\frac{1}{T}\sum_n C(n)\exp\left\{-(2m)^{1/2}\frac{2\pi n}{\alpha\bar{\mu}}\left[(\bar{\mu}\Omega)_{\rm max}{}^{1/2}-\sqrt{E}\right]\right\}\equiv\frac{1}{T}\sum_n C(n)e^{-h(n)}, \tag{15}$$

where T is the bounce period between the adiabatic turning points and the $C(n)$ are appropriate constants. The dominant contribution comes from $n=1$. However, the presence of other values of n shows that there should exist various e-folding times in the decay of the particles, which are integral multiples of the lowest. We present below a comparison of the predictions of the theory $(n=1)$ with the experimental results. The calculations made are approximate in the sense that the field variation is described in all configurations simply by the scale length α^{-1}, characterizing the variation only in the mirror regions through the form of the potential used. The quoted experimental results have been read as accurately as possible from the curves given by the authors. We tabulate in Table I the values of the exponent $h(n)$ in expression (15) for $n=1$ only, evaluated for the corresponding values of the parameters in the experiments. The experimental values of the exponents are obtained from the steepest sections of $\ln\tau$ vs B curves.

A glance at the table shows that the experimental values compare very well with the predictions of the theory even with the approximation made in the calculations. We thus conclude that our model describes very well indeed the average nonadiabatic behavior of particles in the magnetic traps in the slightly nonadiabatic case. The approximation made is worst for the configuration III where the field variation had a more complicated form. The departure is also the largest for this case.

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