the experimental data which was 7.5×10^{-7} at $T=1.39$ K and $v_s=7.8$ cm/sec. At that velocity the vortex line density[14] L was about 7×10^5 cm/cm^3. Using the latter value of $\Delta\bar{\rho}_v/\rho$ we can calculate an upper limit for the fractional change in density per unit length of vortex line per unit volume: $(\Delta\bar{\rho}_v/\rho)L^{-1}\approx10^{-12}$ cm^2. This quantity, having the units of area, can be thought of as representing a region surrounding the axis of the vortex line in which the liquid effectively is either excluded or compressed. Since the residuals obtained from the fit are both positive and negative and are of the order of the resolution of our experiment, both alternatives are possible. Assuming cylindrical symmetry about the vortex axis, a radius corresponding to this area is 25 times larger than the critical radius R_c which is ascribed to the vortex core region using thermodynamic arguments.[15,16] It should be remembered that this is only an upper limit. Any change in the average liquid density due to the structure of the vortex core is probably much less than our estimate.

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Comments on Long-Wavelength Excitations and Structure Functions in the Theory of Liquid ^{4}He at $T=0$

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Polynomial approximations for the liquid structure function $S(k)$ and the energy of an elementary excitation $\epsilon(k)$ are found to contain both odd and even powers of k if the interatomic potential falls off asymptotically as r^{-6}. The proof is given under conditions of low density and weak interaction and hence has only suggestive force in applications to the real superfluid.

A large number of structure functions occur in the microscopic theory of liquid ^{4}He, among them the liquid structure function $S(k)$, the energy of an elementary excitation $\epsilon(k)$, the kinetic structure function $D(k)$, the density-density fluctuation function χ_k, the elementary excitation intensity function $Z(k)$, and the occupation number density $\eta(k)$. This communication concerns polynomial approximations for $S(k)$ and $\epsilon(k)$. Indirectly the argument may suggest the need for flexibility in choosing approximate forms for other structure functions near the origin.

The formula

$$S(k)=\frac{\hbar k}{2ms}\left[1+S_2\frac{\hbar k}{ms}+S_3\left(\frac{\hbar k}{ms}\right)^2+S_4\left(\frac{\hbar k}{ms}\right)^3+\cdots\right] \quad (1)$$

is expected to give a satisfactory representation of the liquid structure function for small values of k ($k \leq 1$ Å^{-1}, $\hbar k/ms \leq 0.664$). Experimental and theoretical information on $S(k)$ and on the energy of elementary excitations $\epsilon(k)$ are consistent with the estimates[1-6] $|S_2/S_3| \ll 1$ and $S_3 \simeq -1.46$. Pines and Woo[6] also obtain an estimate for $S_4 \neq 0$. It appears that polynomial approximations for $S(k)$ may contain both odd and even powers of k. That this is not an altogether trivial statement can be seen by examining the Bogoliubov formalism for the interacting boson system in the limit of low number density ρ and weak interatomic potential $v(r)$. The examination reveals that whether or not Eq. (1) and the corresponding expansion for $\epsilon(k)$ contain both odd and even powers depends on the asymptotic behavior of $v(r)$.

Under the stated conditions the dispersion formula for the energy of an elementary excitation is

$$\epsilon(k) = [(\hbar^2k^2/2m)^2 + 2\rho v_k \hbar^2k^2/2m]^{1/2} = \hbar k s[v_k/v_0 + (\hbar k/2ms)^2]^{1/2}, \tag{2}$$

in which

$$v_k = 4\pi\int_0^\infty \frac{\sin kr}{kr} v(r) r^2 dr$$

and

$$ms^2 = \rho v_0 > 0, \quad v_k > -(\hbar k/2ms)^2 v_0. \tag{3}$$

Also, in agreement with the Bijl-Feynman formula for the energy of an elementary excitation,

$$S(k) = \hbar^2k^2/2m\epsilon(k) = (\hbar k/2ms)[v_k/v_0 + (\hbar k/2ms)^2]^{-1/2}. \tag{4}$$

Equation (4) is given directly by the Bogoliubov formalism if a uniform level of approximation is maintained in evaluating the liquid structure function and diagonal matrix elements of the Hamiltonian (neglect of biquadratic forms in the creation and annihilation operators for excited single-particle levels[7-9]). Equations (2) and (4) and a formula for the ground-state energy identical with that given by the Bogoliubov formalism are all generated by the optimum Jastrow-type trial function in lowest order in the appropriate small parameter. The parameter referred to is $\alpha = 1-g(0)$, a measure of the departure of the radial distribution function $g(r)$ from uniformity.[10]

If $r^n v(r)$ is integrable for all $n > -2$ (as is the case for finite linear combinations of Yukawa and Gaussian functions), the power series for v_k contains only even powers and the corresponding series for $S(k)$ and $\epsilon(k)$ contain only odd powers. Suppose however that $v(r)$ falls off asymptotically as r^{-6} so that

$$\lim_{r\to\infty} r^6 v(r) = W,$$

a constant, positive or negative. This condition with $W < 0$ fits the actual behavior in the real liquid.

The behavior of v_k near the origin can be studied by computing $\Delta_k v_k$ (the Laplacian in $\vec{k}$ space of the function v_k):

$$\Delta_k v_k = -4\pi\int_0^\infty \frac{\sin kr}{kr} v(r) r^4 dr = (\Delta_k v_k)_{k=0} - \frac{4\pi}{k^5}\int_0^\infty \left(\frac{\sin x}{x} - 1\right) v\left(\frac{x}{k}\right) x^4 dx. \tag{4}$$

The integral can be evaluated in the limit of small k. The potential is replaced by the asymptotic form Wk^6/x^6 with the results

$$\Delta_k v_k = (\Delta_k v_k)_{k=0} + \pi^2 Wk + O(k^2) \tag{5}$$

and

$$v_k = v_0 - (2\pi/3)\int_0^\infty v(r) r^4 dr k^2 + (\pi^2/12)Wk^3 + O(k^4). \tag{6}$$

Equations (2) and (6) yield

$$S(k) = \frac{\hbar k}{2ms}\left[1 - \tfrac{1}{8}\left(1 - \frac{\rho m}{\hbar^2}\frac{8\pi}{3}\int_0^\infty v r^4 dr\right)\left(\frac{\hbar k}{ms}\right)^2 - \frac{\pi^2}{24}\frac{m\rho}{\hbar^2}\frac{ms}{\hbar}W\left(\frac{\hbar k}{ms}\right)^3 + O(k^4)\right], \tag{7}$$

and a similar formula for $\epsilon(k)$.

Equation (7) exhibits a mixture of odd and even powers of k in a polynomial approximation for $S(k)$; the same type of mixing is found in $\epsilon(k)$. The mixing occurs because the potential is assumed to fall off asymptotically as r^{-6}. Related expansions involving different mixtures of odd and even powers can be derived for potentials which fall off as r^{-4}, r^{-8}, etc.

These results have only suggestive force in applications to the real superfluid. It is clear, however, that no a priori justification exists for postulating that a polynomial approximation for $\epsilon(k)$ contains only odd powers (although the empirical trial using only odd powers appears to give a satisfactory fit[1]). In this case the apparent absence of even powers may be a problem for the basic theory. Also the possibility that some structure functions [i.e., the elementary excitation intensity function $Z(k)$] are pure cases (without mixing) should be established, if true, by explicit argument.

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Extension of the Lee-Yang Circle Theorem

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We study the zeros of the partition function of a classical spin system and compute a region of the complex fugacity plane where they necessarily lie. We recover the Lee-Yang circle theorem as a special example; we also find that a spin system with finite-range interaction has no phase transition at high temperature.

Asano's recent results[1] have revived interest in the celebrated Lee-Yang circle theorem,[2] by giving it a more conceptual proof. Here an extension of the circle theorem to noncircular regions is proved and applied to problems in statistical mechanics.

(1) Statement of results.– Let P be a complex polynomial in several variables, which is of degree 1 with respect to each, i.e., Λ is a finite set and

$$P(z_\Lambda)=\sum_{X\subset\Lambda} c_X z^X,$$

where $z_\Lambda=(z_x)_{x\in\Lambda}$, and $z^X=\prod_{x\in X} z_x$.

Theorem: Let (Λ_α) be a finite covering of Λ, and for every $x\in\Lambda_\alpha$ let $M_{\alpha x}$ be a closed subset of the complex plane C such that $0\notin M_{\alpha x}$. For each α we assume that the polynomial

$$P_\alpha(z_{\Lambda_\alpha})=\sum_{X\subset\Lambda_\alpha} c_{\alpha X} z^X$$

does not vanish when $z_x\notin -\prod_\alpha(-M_{\alpha x})$, all $x\in\Lambda$. Then the polynomial

$$P(z_\Lambda)=\sum_{X\subset\Lambda} z^X \prod_\alpha c_{\alpha(\Lambda_\alpha\cap X)}$$

does not vanish when[3] $z_x\notin -\prod_\alpha(-M_{\alpha x})$, all $x\in\Lambda$.

The proof is given in Section 2. This result extends a theorem by Lee and Yang,[2] and can be used in the same way to obtain regions free of zeros for polynomials in one variable. More precisely, let the Λ_α be the two-point subsets of Λ: $\Lambda_\alpha=\{x,y\}$ and $c_{\alpha X}=a_{xy}$ when $X=\{x\}$ or $X=\{y\}$, $c_{\alpha X}=1$ when $X=\emptyset$ or $X=\{x,y\}$. For real a_{xy} and $-1\leq a_{xy}\leq 1$ we may take $M_{\alpha x}=\{z\in C:|z|\geq 1\}$; hence[4]

$$Q(\xi)=\sum_{X\subset\Lambda}\xi^{|X|}\prod_{x\in X}\prod_{y\notin X} a_{xy}$$

does not vanish when $|\xi|<1$. By symmetry $Q(\xi)$ does not vanish when $|\xi|>1$, hence the zeros of Q have absolute value 1; this is the Lee-Yang circle theorem.

Let Φ be a real function on $(Z_m)^\nu$ (ν-tuples of integers mod m, "periodic lattice") with $\Phi(x)=\Phi(-x)$, and take $\Lambda=(Z_m)^\nu$. Let again the Λ_α be the two-point subsets of Λ: $\Lambda_\alpha=\{x,y\}$, and write $c_{\alpha X}=\exp[-\beta\Phi(x-y)]$ when $X=\{x,y\}$ and $c_{\alpha X}=1$ when $X=\emptyset$, $\{x\}$, or $\{y\}$. We may then take

$$M_{\alpha x}=\Delta_{xy}{}^\beta,$$

where

$$\Delta_{xy}{}^\beta=\{z\in C:|z+1|\leq(1-e^{\beta\Phi(x-y)})^{1/2}\}\quad\text{for }\Phi(x-y)\leq 0,$$

$$\Delta_{xy}{}^\beta=\{z\in C:|ze^{-\beta\Phi(x-y)}+1|\leq(1-e^{-\beta\Phi(x-y)})^{1/2}\}\quad\text{for }\Phi(x-y)\geq 0,$$

and we find that

$$Q(\xi)=\sum_{X\subset\Lambda}\xi^{|X|}\exp[-\beta\sum_{\{x,y\}\subset X}\Phi(x-y)]$$

can vanish only when

$$\xi\in\Gamma^\beta=-\prod_{y\in Z_m{}^\nu}(-\Delta_{0y}{}^\beta).$$

The region Γ^β is sketched for small and large values of β in Fig. 1. For small β, Γ^β does not