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Force-Free Configuration of a High-Intensity Electron Beam*

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(Received 24 December 1970)

The equilibrium configuration of a neutralized electron beam is calculated, including the self-field (parallel to the beam) generated by the beam itself. The resultant equilibrium configuration allows an arbitrary current to be carried by the beam, the motion of electrons being unimpeded by the self-magnetic field. Electrons in this configuration travel almost parallel to the magnetic field; thus, the synchrotron radiation can be minimized. It is suggested that this type of configuration opens up the possibility of low-loss electron coils and, in particular, is useful for fusion reactors.

The limitation for relativistic electron-beam currents (space-charge neutralized) was calculated previously from the condition that the Larmor radius of electrons in the self-field should not exceed the radius of the beam, a. The condition, in terms of magnetohydrodynamics (MHD), is that the equation

$$\vec{J}\times\vec{B}=\nabla W_\perp \tag{1}$$

must be satisfied. Here, $\vec{J}=(0,0,J_z)$ is the beam current density, and $\vec{B}=(0,B_\theta,0)$ is the self-field produced by J; $W_\perp$ is the perpendicular energy density of the electrons, which is variable in order to satisfy Eq. (1) but obviously limited from above by the total energy. From this one gets

$$I^2\sim 4\pi^2a^2W_\perp/\mu_0\le 4\pi^2a^2W/\mu_0, \tag{2}$$

where I is the beam current (we assume that there is no return current present). Therefore,

$$I\le 4\pi^2a^2W/I\mu_0\approx 10^4\gamma\beta\ \mathrm{A}\equiv I_A; \tag{3}$$

here the substitution $I\approx\pi a^2nve$ was made (mks), $\beta=v/c$, and $\gamma=(1-\beta^2)^{-1/2}$. A more precise calculation yields $I_A\approx 1.7\times10^4\gamma\beta$ A.[1,2]

By looking at Eq. (1) we immediately notice that there is no reason why $\vec{J}$ is restricted to J_z. Indeed, J_θ can flow in the system to create a component B_z. It is known that with an external field imposed,[2] Eq. (1) can have an I that is beyond the limit imposed by the relation (3). Here we ask, is it possible to arrive at an equilibrium condition for which Eq. (1) can be satisfied, even in the absence of an external magnetic field, for arbitrary current strength I without any abnormal configuration such as that of a hollowed-out beam[2] [the hollowed-out beam still satisfying Eq. (1) with $J_\theta=0$]? The answer is affirmative. Let us, for a moment, ignore the right-hand side of Eq. (1). Then we get the well-known force-free field system and the solution exists in the infinite-beam system. (The question for the finite-beam system is different. Since our interest is the self-consistent field solution, with possible application to a relativistic electron ring, the infinite system is treated here.) We treat the case where background infinite-mass ions are present to neutralize the space change.

The inclusion of the right-hand side, because of the centrifugal force acting on electrons, is now treated. We assume that all the electrons have $v_r=0$. Then each electron must satisfy the relation (m being the relativistic mass of electron)

$$\vec{e}_r m v_\theta^2/r=e(\vec{v}\times\vec{B}), \tag{4}$$

where $\vec{e}_r$ is the unit vector in the radial direction. We assume that there is no velocity dispersion; that is,

$$\vec{J}=-ne\vec{v}. \tag{5}$$

Thus, instead of Eq. (1) we get

$$(\vec{J}\times\vec{B})_r=-mJ_\theta^2/ne^2r. \tag{6}$$

Or, using $\mu_0\vec{J}=\nabla\times\vec{B}$ and noting that $\partial\vec{B}/\partial\theta=\partial\vec{B}/\partial z=0$, we arrive at

$$\frac{1}{2}\frac{dB_z^2}{dr}+\frac{1}{r}\frac{d(rB_\theta)}{dr}B_\theta=\frac{m}{\mu_0nre^2}\left(\frac{dB_z}{dr}\right)^2. \tag{7}$$

We take

$$B_z = B_1 + (1 - r^2/a^2)B_2, \quad r \le a, \tag{8}$$

$$n = n_0 = \text{const}, \quad r \le a, \tag{9}$$

$$n = 0 \text{ and } B_z = B_1, \quad r > a, \tag{10}$$

where B_1 and B_2 are arbitrary constants.

In highly relativistic cases it is more pertinent to let $|\vec{v}| = \text{const}$ and let n be a function of r. In a case where I exceeds I_A, however, the density variation is less than ~7% (if $B_1/B_2 \ge 0$). Therefore, we present here a simpler case in which $n = n_0$ and $|\vec{v}|$ is a function of r (again the variation of v is less than ~7%, if $B_1/B_2 \ge 0$). Then, for $r \le a$,

$$-B_z \frac{2r}{a^2} B_2 + \frac{1}{2}\frac{1}{r^2}\frac{d(rB_\theta)^2}{dr} = \frac{m}{ne^2}\frac{1}{\mu_0}\frac{4r}{a^4} B_z{}^2, \tag{11}$$

or

$$\frac{d(rB_\theta)^2}{dr} = (B_1 + B_2)\frac{4r^3}{a^2}B_2 - \frac{4r^5}{a^4}B_2{}^2 + \frac{m}{ne^2\mu_0}B_2{}^2\frac{8r^3}{a^4}. \tag{12}$$

Obviously, $rB_\theta = 0$ at $r = 0$ (otherwise there is a singularity at $r = 0$ [3]); hence,

$$B_\theta{}^2 = (B_1 + B_2)\frac{r^2}{a^2}B_2 - \frac{2}{3}\frac{r^4}{a^4}B_2{}^2 + \frac{m}{ne^2\mu_0}B_2{}^2\frac{2r^2}{a^4}. \tag{13}$$

This is the general solution for $B_\theta{}^2$ for given B_1 and B_2. For $r \ge a$, $B_\theta = B_0(a/r)$, where B_0 is the value of B_θ at $r = a$. The interesting possibility lies in $B_1 = 0$. Then the total beam current is given by

$$I^2 = \frac{4\pi^2a^2}{\mu_0{}^2}\left(\frac{1}{3} + \frac{2m}{a^2ne^2\mu_0}\right)B_2{}^2 = \frac{4\pi^2a^2}{\mu_0{}^2}\left(\frac{1}{3} + \frac{2c^2}{\omega_{pe}{}^2a^2}\right)B_2{}^2 = \frac{4\pi^2a^2}{\mu_0{}^2}\left(\frac{1}{3} + \frac{I_A}{\pi\beta cena^2}\right)B_2{}^2. \tag{14}$$

Since $\pi\beta cena^2$ is approximately I, it follows if $I \gg I_A$, Eq. (14) reduces to $I^2 \sim (4\pi^2a^2/3\mu_0{}^2)B_2{}^2$. That is, arbitrarily high current can flow in the beam and the current distribution will be close to the force-free condition

$$\vec{J} \times \vec{B} = 0. \tag{15}$$

This configuration has another interesting property. If $I \gg I_A$, the electrons follow the magnetic field line very closely. The pitch length l of the magnetic field line around the center axis of the beam can be calculated easily. The pitch distance is ($I \gg I_A$)

$$l = \int \frac{ds\, B_z}{B_\theta} = 2\pi r \frac{B_1 + [1 - (r^2/a^2)]B_2}{B_\theta}.$$

The pitch length and curvature of the line versus r, for $B_1 = 0$, are plotted in Fig. 1. The electron orbit, at $r = a$, is a helix (unlike the magnetic field line). The pitch length l_e for the electron is

$$l_e = \int ds \frac{J_z}{J_\theta} = 2\pi r \frac{B_1 + B_2(1 - r^2/a^2) + mB_2/(ne^2\mu_0 a^2)}{B_\theta};$$

in particular, at $r = a$ and $B_1 = 0$, we get

$$\frac{l_e}{2\pi a} \approx \sqrt{3}\frac{m}{ne^2\mu_0 a^2} = \frac{\sqrt{3}}{2}\frac{I_A}{\pi\beta cena^2}.$$

This is small but finite. It can be shown that $l_e/2\pi a$ never goes to 0 in $0 \ge r \ge a$.

The conclusion drawn from this is that all electrons move in a spiral path, with the radius of curvature greater than a. Since the current can exceed I_A by a large factor, it follows that the radius of the curvature of the individual electron path exceeds the maximum Larmor radius of the individual electron at the same energy by a factor of I/I_A. That means, in this particular configuration, that the synchrotron radiation is reduced by a large factor. Even if we start with a considerable amount of perpendicular energy ($v_r \ne 0$), we expect the perpendicular energy to be radiated away quickly by the synchrotron radiation. Thus, this configuration is most likely to be reached if the beam is to last an appreciable

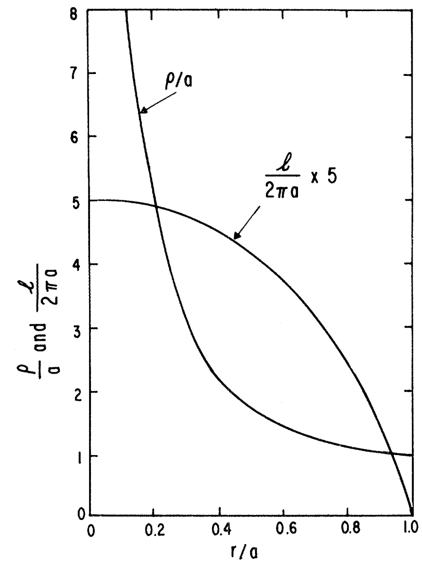

FIG. 1. Radius of curvature and pitch length of electrons versus the radius of the electron beam.

time. There is a hope, therefore, that the lifetime of an electron ring, due to radiation, is much longer than previously expected.

The experimental observation of a large current in relativistic electron beam experiments is possibly due to this mechanism. The measurement of the magnetic field parallel to the beam should test this possibility.

We now investigate the orbit of electrons in a force-free electron ring. Since the inertia term in Eq. (6) is small if $I \gg I_A$, we neglect this, and solve, instead, $\vec{J}\times\vec{B}=0$ for the toroidal case. The general virial theorem states that a vertical field B_v must be imposed for the solution. However, with the imposition of B_v there is an equilibrium solution, governed by the equation[4]

$$\frac{\partial^2\psi}{\partial\rho^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}=-\frac{1}{2}\frac{\partial}{\partial\psi}\kappa^2,$$

with $\psi=\rho A_\varphi$ and $\kappa=\rho B_\varphi$ in cylindrical coordinates (ρ, φ, z); the ring minor axis is $z=0$ and $\rho=R$. Then the solution for which $a/R \ll 1$ is very much like that of the straight cylinder. Thus, the radius of the curvature of the electron still remains $\sim a$ or more, if the external toroidal field is zero. By applying a strong toroidal field so that the rotational transform can be made 2π or even less, we can make the radius of the curvature of the electron even larger, up to R. Therefore, if we are to reduce the radiation to keep the relativistic electron ring for a long time, it is better to increase the toroidal magnetic field! For Astron[5] (with or without toroidal field) with the 30-MeV electron ring of major radius 1 m, minor radius 30 cm, the equivalent magnetic field which causes the same amount of radiation is between $B=4$ kG (for 30 cm) and $B=1$ kG (for 1 m). Thus, the loss due to radiation can be minimized, and an electron beam may be used in Astron for a fusion reactor.

It has been demonstrated that an equilibrium configuration exists for an electron beam, even when $I > I_A$. It is proposed that this may be the explanation of the observed high-current relativistic beam production. Also, this equilibrium configuration is such that electrons flow force free, almost parallel to $\vec{B}$. Therefore, the radius of curvature of an individual electron is greatly increased over the Larmor radius. Hence, it is possible that the synchrotron radiation can be reduced. The implication for both the ring accelerator and Astron is that the lifetime of the relativistic electron ring may be much larger than hitherto estimated from the radiation energy loss due to the self-magnetic field. Furthermore, in the future it may be possible to construct a low-resistance, high-current coil by means of a relativistic electron beam in situations where superconducting coils are not suitable.

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*Work supported by U. S. Atomic Energy Commission under Contract No. AT(30-1)-1238.

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