Final States of Gravitational Collapse*

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We examine all of the black-hole geometries which can be analytically developed in terms of a parameter from the Schwarzschild geometry. It is shown that this analytic family is completely spanned by the Kerr-Newman space-times with $e^2+a^2<m^2$, where e, a, and m denote charge, specific angular momentum, and mass. If general (nonspherical) gravitational collapse produces black holes and if analytic variation of the initial conditions of gravitational collapse causes analytic variation of the final space-time geometry of the black holes produced by the collapse, this result implies that the generic final state of gravitational collapse is a Kerr-Newman black hole, fully specified by its mass, angular momentum, and charge.

The only known solutions of the Einstein field equations which describe black holes are the Kerr-Newman solutions,[1-4] whose space-time metric and electromagnetic field tensor may be written, respectively, as follows (in geometrical units $c=G=1$):

$$ds^2=-\frac{r^2+a^2+e^2-2mr}{r^2+a^2\cos^2\theta}(dt-a\sin^2\theta d\varphi)^2+\frac{\sin^2\theta}{r^2+a^2\cos^2\theta}[adt-(r^2+a^2)d\varphi]^2$$
$$+(r^2+a^2\cos^2\theta)\left(\frac{dr^2}{r^2+a^2+e^2-2mr}+d\theta^2\right), \tag{1}$$

and

$$F=\frac{2e}{(r^2+a^2\cos^2\theta)^2}(r^2-a^2\cos^2\theta)dr\wedge(dt-a\sin^2\theta d\varphi)-\frac{4ear\sin\theta\cos\theta}{(r^2+a^2\cos^2\theta)^2}d\theta\wedge[adt-(r^2+a^2)d\varphi]. \tag{2}$$

The Kerr-Newman black holes depend on only three parameters, m, a, and e, representing, respectively, mass, angular momentum per unit mass, and charge.

This Letter reports a new result which indicates that the generic final state of gravitational collapse is a Kerr-Newman black hole.

It is well established that stars with mass greater than a certain critical mass (approximately one solar mass) cannot complete their thermonuclear evolution without undergoing gravitational collapse. For a star of perfect spherical symmetry, complete collapse must result in a Schwarzschild black hole (or in a Reissner-Nordstrom black hole if the star has a net charge). A full treatment of the dynamics of nonspherical collapse[5-7] remains a task for the future, but it is natural to believe that the intense gravitational fields occurring as the nonspherically symmetric star collapses will result, as in the spherically symmetric case, in self-closure of the star (i.e., the enclosing of the star within an event horizon) while the star is still of finite size and has finite density. If this is the case, then it is natural to expect the region exterior to the horizon in a general collapse situation to retain the following three properties that it has in spherically symmetric collapse:

(1) This region should be vacuum except possibly for electromagnetic fields (this will be called an EV region) since all but a negligible fraction of the matter initially present should fall through the horizon or be ejected to large distances.

(2) The horizon and exterior region should be nonsingular. This is because the asymmetries should not have such a drastic effect before self-closure occurs (while the star is still of finite density) that they create singularities; but after self-closure occurs, the star is completely cut off from the exterior region and therefore no drastic changes should occur in this region after self-closure occurs either.

(3) The space-time metric should "settle down" and approach a final-state metric which is pseudostationary (i.e., has a Killing vector which is timelike at spatial infinity) because only a finite amount of gravitational radiation can be emitted in the collapse process.

Furthermore, it is natural to believe that if the initial conditions of gravitational collapse are smoothly (i.e., analytically) varied, then the space-time geometry of the exterior regions of the black holes produced by the collapse will also vary smoothly since infinitesimal changes in the initial conditions should not result in drastic changes in the final exterior region. (However, it should be pointed out that this assumption is not entirely trivial; as the initial conditions are smoothly varied, some instability might suddenly occur in the collapse process, resulting in nonsmooth variation of the final states. In fact, the example of Newtonian ellipsoidal equilibrium configurations of homogeneous, ideal fluids[8] —many of which cannot be obtained by analytically deforming the spherically symmetric configuration through other equilibrium configurations —illustrates the possibility that this could happen.)

Now, imagine varying the initial conditions of gravitational collapse (e.g., initial density, rotation, etc.) smoothly with a parameter α, with $\alpha = 0$ corresponding to perfect spherical symmetry and no net charge; and consider, for each α, the final-state metric $g(\alpha)$ and final electromagnetic field tensor $F(\alpha)$ resulting from the collapse. If the above stated natural assumptions concerning collapse are correct, $g(\alpha)$ and $F(\alpha)$ will satisfy the following five properties:

(I) Each pair $g(\alpha), F(\alpha)$ is a solution of the Einstein-Maxwell equations [i.e., $g(\alpha)$ satisfies Einstein's equations with energy-momentum tensor due to $F(\alpha)$; $F(\alpha)$ satisfies Maxwell's equations in the geometry $g(\alpha)$]. $g(0)$ is the Schwarzschild metric and $F(0)=0$.

(II) Each $g(\alpha)$ is asymptotically flat and pseudostationary. This requires that for each α there exist "Schwarzschild-like coordinates" t, r, θ, and φ having the following properties: (a) $\partial/\partial t$ is a Killing vector which is timelike at spatial infinity; and (b) the metric is asymptotically Minkowskian and the field-tensor components are $O(1/r^2)$ as $r\to\infty$ in the coordinates t, x, y, and z, where $x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$, and $z=r\cos\theta$.

(III) The exterior region (i.e., the domain of outer communications) and horizon of each $g(\alpha)$ is nonsingular. (The domain of outer communications is defined as the set of points lying on timelike curves which escape to spatial infinity in both the future and past directions; the horizon is defined as its boundary.) This requires the existence of "analytic coordinates" (i.e., "generalized Kruskal coordinates") which analytically cover the exterior region and horizon of $g(\alpha)$. (The Schwarzschild-like coordinates need not analytically cover this entire region, e.g., they may break down at the horizon.)

(IV) The metric and field tensor components in the analytic coordinate system vary analytically with α.

(V) The transformation from the Schwarzschild-like coordinates to the analytic coordinates varies analytically with α.

The analytic family of EV black-hole metrics containing the Schwarzschild metrics is defined as the set of all space-times belonging to some one-parameter family $g(\alpha)$ which satisfies (I)-(V). Since the initial conditions of general gravitational collapse can be obtained by analytic (in α) variation of the initial conditions of spherical collapse, this analytic family must coincide with the possible final state of collapse if the above stated natural beliefs concerning collapse are correct. Theorems proven by Israel[9,10] show that the Reissner-Nordstrom metrics are the only static (as opposed to merely pseudostationary) black holes which have closed, simply connected surfaces of constant g_{00}. Carter[11] has shown that the class of axially symmetric "simple black-hole exterior solutions" with no electromagnetic fields consists of a discrete set of one-parameter or two-parameter "continuous families" [i.e., families such that any two elements can be connected by a piecewise analytic (as opposed to analytic) curve $g(\alpha)$ consisting of only elements of the family]. Thus, Israel's theorems indicate that the Reissner-Nordstrom solutions with $|e|<m$ are the only static members of the analytic family defined above, while Carter's theorem shows that the Kerr space-times with $|a|<m$ comprise the only axially symmetric subfamily of this analytic family which have no electromagnetic fields and contain the Schwarzschild metrics. The new theorem is the following:

Theorem.—The analytic family of EV black-hole metrics containing the Schwarzschild metrics is completely spanned by the Kerr-Newman space-times with $e^2+a^2<m^2$, i.e., any one-parameter family $g(\alpha), F(\alpha)$ satisfying conditions (I)-(V) is composed only of Kerr-Newman solutions, Eqs. (1) and (2).

That the Kerr-Newman space-times with $e^2+a^2<m^2$ belong to this analytic family (but not those with $e^2+a^2=m^2$ where the topology of the horizon changes or those with $e^2+a^2>m^2$ where there is no horizon at all) is easily verified from the well-known properties of these metrics.[4] The nontrivial part of the proof consists of showing that there are no other members, i.e., that any one parameter family $g(\alpha)$ satisfying (I)-(V) is composed only of Kerr-Newman metrics. The basic idea of the proof is as follows (for simplicity, only the pure vacuum case, i.e., no electromagnetic fields, is discussed).

We examine all of the derivatives $g^{(n)}=d^ng/d\alpha^n|_{\alpha=0}$ of $g(\alpha)$ with respect to α at $\alpha=0$ and translate the conditions (I)-(V) into conditions on the $g^{(n)}$. We find that $g^{(1)}$ must be a time-independent solution of the linearized Einstein equations in the Schwarzschild background, that its components in the coordinate system t, x, y, and z must be $O(1/r)$ as $r\rightarrow\infty$, and that its components in the Kruskal coordinates of the background Schwarzschild metric must be finite at the horizon up to a gauge transformation (i.e., there must exist a gauge transformation f^μ such that the Kruskal components $g_{\mu\nu}{}^{(1)}+f_{\mu;\nu}+f_{\nu;\mu}$ are finite at the horizon). Vishveshwara's[12] analysis of stationary perturbations of the Schwarzschild metric effectively shows that to first order in α, $g(\alpha)$ must coincide with a family composed only of Kerr metrics. The generalization of this statement to all orders is made possible by the following crucial fact: The nth-order equations and conditions involve nth-order quantities in precisely the same manner as the first-order equations involve first-order quantities. Hence, if $g(\alpha)$ and $g'(\alpha)$ agree in all orders $j\leq n-1$, we find that $g^{(n)}-g'^{(n)}$ must satisfy the same equations and conditions as $g^{(1)}$, so that, given all $g^{(j)}$ with $j\leq n-1$, $g^{(n)}$ is determined up to the same (highly restricted) arbitrariness as exists for $g^{(1)}$. Thus, we inductively obtain a uniqueness theorem for the $g^{(n)}$, from which it follows that $g(\alpha)$ is composed of only Kerr metrics.

Thus, the above theorem plus the assumptions that (1) gravitational collapse produces black holes and (2) analytic variation of the initial conditions of gravitational collapse causes analytic variation of the black-hole space-time geometry resulting from the collapse, lead one to conclude that the generic final state of gravitational collapse is a Kerr-Newman black hole with $e^2+a^2<m^2$. That only final states with $e^2+a^2<m^2$ exist is not very surprising since from Newtonian considerations it would be expected that, if a body has $e^2+a^2>m^2$, the electrostatic and centrifugal repulsive forces ($\sim e^2/r+ma^2/r^3$) would dominate over the gravitational attraction ($\sim m^2/r^2$) before a horizon could be formed at $r\sim m$, so it is quite reasonable that a body with $e^2+a^2>m^2$ cannot collapse. (On the other hand, small charge and rotation cannot prevent collapse.[7]) Much more surprising is the conclusion that the final state of a collapsed object is completely determined by its mass, angular momentum, and charge, but recent work of de la Cruz, Chase, and Israel,[13] Price,[14] and Cohen and Wald[15] substantiates this conclusion for small deviations from spherical symmetry.

One of the most significant astrophysical consequences of the above conclusion is that a collapsed object cannot have an electromagnetic field unless it is charged [in which case its field must still be of the very special Kerr-Newman form, Eq. (2)]. The net charge of an uncollapsed object must be very small, for if it were not, excess protons or electrons would fly off from the star under the influence of electrostatic repulsion. Moreover, no mechanism is known by which charge separation could occur during collapse. Consequently, it appears that collapsed objects cannot play a direct role in phenomena requiring a magnetic field, such as synchrotron radiation.

Details of the proof of the theorem discussed above will be submitted for publication elsewhere.

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