algebra.

[5]R. A. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971).

[6]For a review of the theory and a summary of applications, see R. A. Brandt and G. Preparata, in Proceedings of the 1971 Coral Gables Conference on Symmetry Principles at High Energies (to be published).

[7]S. Fubini and G. Furlan, Physics (Long Is. City, N.Y.) 1, 229 (1965).

[8]R. A. Brandt and G. Preparata, Ann. Phys. (New York) 61, 119 (1970).

[9]M. Gell-Mann, R. J. Oakes, and B. Renner [Phys. Rev. 175, 2195 (1968)] have used a different set of assumptions in connection with the $(\underline{3},\underline{3}^*)\oplus(\underline{3}^*,\underline{3})$ model and have found $c/\sqrt{2}\simeq -1$. The recent accurate numerical evaluation of $T^{\pi N(+)}(\nu=0,\ \nu_B=0)$ by T. P. Cheng and R. F. Dashen [Phys. Rev. Lett. 26, 594 (1971)], however, makes it highly unlikely that both these different assumptions and the $(\underline{3},\underline{3}^*)\oplus(\underline{3}^*,\underline{3})$ model are correct.

[10]R. A. Brandt and G. Preparata, Phys. Rev. Lett. 25, 1530 (1970).

[11]Although our language is different, we are adopting here the philosophy of Weinberg (Ref. 2) in making a sharp distinction between the operator algebras and the symmetry properties of the states.

[12]J. D. Bjorken, Phys. Rev. 148, 1467 (1966).

[13]See Ref. 2 for a discussion of some previous attempts to calculate these quantities.

[14]For most of the cases considered in Ref. 8, we used Regge theory to determine which amplitudes extrapolated smoothly off-shell. For the K_{l3} amplitudes, Regge theory is inapplicable; but we argued that the off-shell extrapolation defined by $\mathfrak{D}$ was smooth and that the one defined by V_μ was not. This led to $f_+(m^2)+f_-(m^2)\simeq 0.3f_K/f_\pi$. Now, however, we need not make such assumptions since we can reliably compute off-shell corrections. Our result is that neither extrapolation is smooth.

Description of High-Energy Scattering

B. B. Deo and M. K. Parida
Physics Department, Utkal University, Bhubaneswar-4, Orissa, India
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A model of high-energy scattering is developed which incorporates assumed analytic properties. The approximations involved in the phenomenological fits of Orear and Krisch are discussed. An improved formula is suggested which has correct analytic properties and reproduces the Orear and Krisch fits in appropriate limits.

At present there is no general theory available for explaining high-energy hadron collision cross sections over the entire angular range. For proton-proton scattering there exist at least two excellent phenomenological fits,[1] the Orear[2] formula

$$S\,d\sigma/d\Omega = A e^{-(p\sin\theta)/b} \tag{1}$$

and the Krisch[3] formula

$$\begin{aligned} d\sigma/dt &= \textstyle\sum_i A_i \exp(-\alpha_i \beta^2 p^2) \\ &= \textstyle\sum_i A_i \exp(-\alpha_i ut/s), \end{aligned} \tag{2}$$

with $A_1=90$, $A_2=0.74$, $A_3=0.0029$, $\alpha_1=10$, $\alpha_2=3.45$, and $\alpha_3=1.45$. Here s, t, and u are the usual Mandelstam variables, θ is the c.m. scattering angle, and p is the c.m. momentum. The Orear fit is good for large angles around $\theta\simeq 90°$, but deviates substantially for small angles. A single exponential[4] in t fits the data for small t, i.e., for θ close to 0°, and is quite bad for larger angles. It is worth mentioning that a fit which combines both the Krisch and Orear type of exponentials,

$$d\sigma/d\Omega = A\ \exp(\alpha p_\perp + \beta p_\perp^2), \tag{3}$$

had been proposed by Narayan and Sarma[5] as early as 1964.

So far, no attempts have been made to propose phenomenological formulas which have the correct analytic properties of scattering cross sections. Since the analytic properties are due to the nature of forces responsible for scattering of hadrons, a formula satisfying the requirements of analyticity could provide better representations for scattering data. This was first pointed out by Cutkosky and Deo[6] and Cutkosky[7] and also explicitly demonstrated by them[8] and Chou.[9] They have been able to construct polynomial expansions where each term has the assumed analytic property and more importantly, the series converges very rapidly. We propose to extend some of these ideas to the high-energy scattering for all angles in the diffraction region. At these energies, resonances have ceased to contribute appreciably so that all relevant functions and the discontinuities are fairly smooth functions of chosen variables.

We ignore complications due to spin and consider equal-mass hadron scattering.[10] A function $f(s, t)$ is chosen which is proportional to the differential scattering cross section $d\sigma/d\Omega$. For equal-mass scattering $f(s, t) = f(s, u)$, i.e., the differential scattering cross section is symmetrical about $\theta = 90°$. This enables one to treat $f(s, t)$ conveniently as a function of $\cos^2\theta$ rather than $\cos\theta$.

$f(s, t)$, being proportional to $d\sigma/d\Omega$, has the same analytic properties as the scattering amplitude $A(s, t)$ for a given s, since $s d\sigma/d\Omega = |A(s, t)|^2$. So we assume $f(s, t)$ to be an analytic function in the cut $\cos\theta$ plane with *cuts* from $-is$ to $-x_{\mathrm{el}}$ and x_{el} to ∞. There are also a large number of inelastic cuts, the nearest one starting from $\pm x_{\mathrm{in}}$.

Cutkosky and Deo[6] and Ciulli[11] mapped the symmetrical cut plane into the inside of an ellipse or a region enclosed by two circles. Unfortunately at high energies the cuts come close to ± 1 and the ellipse shrinks onto the physical region making a Legendre expansion unreliable.

Cutkosky[12] has developed a statistical model of high-energy scattering processes which also incorporates assumed analytic properties. He has concluded that the absence of Ericson fluctuations *may* imply a more extended domain of analyticity than the small Lehmann ellipse so that the partial-wave amplitudes are strongly correlated to conform to analyticity requirements as shown by phase-shift analysis.[8,9] Following up this line of investigation, we propose to map the entire cut plane into the inside of a parabola so as to (i) effectively increase the region of analyticity and (ii) hold the cuts away from the physical region.

Our attempt is not to find a curve with the best fit to the experimental data but to suggest a simple analytic formula for a curve which would give a very good fit to the experimental data. To this end, we wish to expand the given function $f(s, t)$ in a series of classical orthogonal polynomials. We assume that $f(s, t)$ is square integrable, i.e.,

$$\int_{\Gamma(\xi)} |f(s, \xi)|^2 d\xi < \infty, \tag{4}$$

where $\xi = \xi(x)$ is a variable suitably constructed and $\Gamma(\xi)$ is the physical domain of $f(s, \xi)$. The expansion[13]

$$f(s, \xi) = [\omega(\xi)]^{1/2} \textstyle\sum_n C_n(s) P_n(\xi) \tag{5}$$

converges in certain regions of the ξ plane, $\omega(\xi)$ being the weight function corresponding to the orthogonal polynomial $P_n(\xi)$. When ξ lies between -1 and $+1$, the orthogonal polynomials could be the Legendre polynomials with $\omega(\xi) = 1$; the region of convergence is the largest unifocal ellipse free from singularities. The essential idea of Ref. 6 was to map the x plane such that the entire cut plane was within the ellipse of convergence in the ξ plane. If, in a physical situation, the physical region extends from $-\infty$ to $+\infty$, the corresponding orthogonal polynomials suitable for expansion are the Hermite polynomials with $\omega(\xi) = \exp(-\xi^2)$, the region of convergence being a strip around the real axis. The entire region of analyticity should then be mapped into the inside of the strip in the ξ plane.

For the high-energy problem we have chosen to consider here, the physical region will be conveniently formed so as to extend from zero to infinity. The weight function is then $\exp(-\xi)$ and Laguerre polynomials $L_n(\xi)$ are the corresponding classical polynomials suitable for expansion. The region of convergence of the Laguerre-polynomial expansion is a parabola with the origin as focus. So we will obtain ξ by suitable conformal mappings such that the entire cut $\cos\theta$ plane is mapped into the inside of the parabola in the ξ plane. In such a case we expect

$$f(s, t) = \exp(-\xi/2) \textstyle\sum_n C_n L_n(\xi) \tag{6}$$

to converge very rapidly. In the sense that no region of analyticity is left out of the region of convergence, the expansion (6) may also be considered optimal. Phenomenological fits suggest very strongly that the gross features of high-energy scattering in the diffraction region are described by a single term, in which event the first term would provide an adequate representation of high-energy data. It is useful to introduce t_{el} and t_{in} through the relations $x_{\mathrm{el}} = 1 + t_{\mathrm{el}}/2p^2$ and $x_{\mathrm{in}} = 1 + t_{\mathrm{in}}/2p^2$. Figure 1(a) shows the cut $x = \cos\theta$ plane, and the simple mapping[14]

$$y = (1 - x^2)/(x^2_{\mathrm{el}} - 1) \tag{7}$$

is shown in Fig. 1(b). The cuts now extend from -1 to $-\infty$ and the physical region from 0 to y_∞, where $y_\infty = (p^2/t_{\mathrm{el}})(1 + t_{\mathrm{el}}/4p^2)^{-1}$, tends to infinity with p^2.

Next we open out the elastic two-particle cut[15] by the mapping

$$\omega = [(x_{\mathrm{el}}^2 - x^2)^{1/2} - (x_{\mathrm{el}}^2 - 1)^{1/2}](x_{\mathrm{el}}^2 - 1)^{-1/2}. \tag{8}$$

This is shown in Fig. 1(c). The physical region

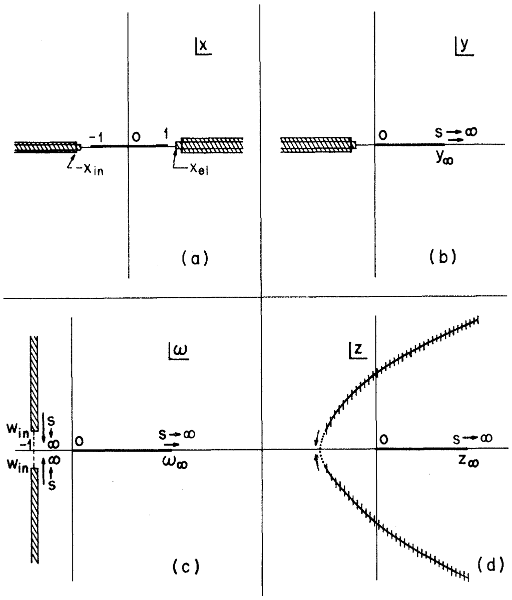

FIG. 1. The bold lines indicate the physical region and the direction of the arrows show the elongation of the physical region with energy. The dotted lines in (c) and (d) are the inelastic cuts which shrink to the real axis as energy increases.

extends from $\omega=0$ to $\omega=\omega_\infty$, where $\omega_\infty=[x_{\text{el}}-(x_{\text{el}}^2-1)^{1/2}](x_{\text{el}}^2-1)^{-1/2}$ tends to infinity as $pt_{\text{el}}^{-1/2}$ for large energy. The inelastic uts extend from $\omega_{\text{in}}^{\pm}=-1\pm i(x_{\text{in}}^2-x_{\text{el}}^2)^{1/2}(x_{\text{el}}^2-1)^{-1/2}$ to $-1\pm i\infty$.

The inelastic cuts are then mapped conformally to form the boundary of a parabola with the origin as the focus. The mapping is complicated. However, as $p^2\to\infty$ we have $x_{\text{in}}^2-x_{\text{el}}^2\to 0$ and $\omega_{\text{in}}^{\pm}\to -1$, so that the inelastic cuts meet at -1 ± 0, extending from $(-1-i\infty)$ to $(-1+i\infty)$. A simple conformal transformation,

$$Z=+[\operatorname{inv}\cosh(1+\omega)]^2, \tag{9}$$

maps the analytic ω plane into the interior of the parabola with the origin as the focus as shown in Fig. 1(d). The physical region lies on the real axis from 0 to Z_∞ and Z_∞ tends to infinity as $\ln(2p/t_{\text{el}}^{1/2})$.

We therefore suggest to the experimentalists to fit their high-energy data by the expression embodying correct analyticity, namely,[16]

$$d\sigma/dt=A\exp(-\alpha Z/2). \tag{10}$$

It is easily checked that for $|t|\ll 0.8\ \text{BeV}^{-2}$, $Z\simeq ut/4p^2t_{\text{el}}$; and for finite angles but high energies, $Z\to(p\sin\theta)/t_{\text{el}}$. Thus our proposed formula (10) is like $\exp(at)$ for very small values of t but for large angles it reduced to the Orear formula, $\exp[(-p\sin\theta)/b]$, with $b=\alpha/2t_{\text{el}}$. Krisch and Orear fits are obtained by ignoring the cut structure and using the variables of Eqs. (7) and (8), respectively, in the exponential form.

We make no attempt to explain the dips observed for larger t. To explain such features higher terms in our expansion may have to be taken, provided perhaps that resonances are not responsible for the dips.

We are thankful to Professor R. E. Cutkosky for pointing out the importance of the problem and also for going through the manuscript.

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[2] J. Orear, Phys. Lett. 13, 190 (1964).

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[6] R. E. Cutkosky and B. B. Deo, Phys. Rev. Lett. 20, 1272 (1968), and Phys. Rev. 174, 1859 (1968).

[7] R. E. Cutkosky, Ann. Phys. (New York) 54, 350 (1969).

[8] R. E. Cutkosky and B. B. Deo, Phys. Rev. D 1, 2547 (1971), and Ref. 6.

[9] Y. A. Chao, Phys. Rev. Lett. 25, 309 (1970).

[10] For unequal-mass scattering, the asymmetrical cut plane is to be symmetrized by the transformation suggested in Ref. 6. One cannot use the symmetry properties. However, it may be possible to parametrize $d\sigma/dt=f(\cos^2\theta)+(\cos\theta)g(\cos^2\theta)$.

[11] S. Ciulli, Nuovo Cimento 61A, 787 (1969).

[12] R. E. Cutkosky, Nucl. Phys. B13, 351 (1969).

[13] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill, New York, 1953), Vol. II. See especially Chap. X, Sects. 10.2, 10.15, and 10.19.

[14] Because of symmetry for $x\to -x$ in equal-mass scattering, the first and third quadrants and second and fourth quadrants overlap in an x^2 transformation. In unequal-mass scattering this may have to be suitably modified.

[15] More elegant conformal maps have been devised and used by R. E. Cutkosky and C. C. Shih, to be published and private communications.

[16] It is perhaps necessary to show that Eq. (4) is satisfied with $d\sigma/dt\sim f(s,\xi)$. In the diffraction region $f(s,\xi)<A=(d\sigma/dt)_{t=0}$. So the integral is less than A^2 times the length of ξ, the physical region. Pomeranchukon exchange gives a constant A whereas the experiments of Ref. 4 suggest $A\sim p^{-0.2}$. We have, however, the option of renormalizing $d\sigma/dt$ by multiplying by a suitable function of s in our fixed-energy analysis.