Phonon Excitations in Liquid He II

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Recent experimental data on He II have been analyzed in order to test various analytic properties of the phonon dispersion curve. In particular $[\epsilon(p)/p]^2$, where $\epsilon(p)$ is the energy of an elementary excitation, seems to be well represented by a power expansion in p up to $p=2.6$ Å^{-1}.

A series of recent investigations on the structure of the He II phonon dispersion curve has suggested a number of drastic changes in the long-wavelength region. These new proposals fall into two rather separate categories. The first[1-3] points out the existing experimental evidence for an upward concavity of the phonon curve at small momenta as gathered from specific-heat, sound-attenuation, and perhaps neutron-inelastic-scattering measurements. The second[4] proposes, on theoretical grounds, the inclusion of odd-power terms in the local expansion for the energy $\epsilon(p)$ of an elementary excitation,

$$\epsilon(p)/c\hbar p=(1-\alpha_1 p-\alpha_2 p^2-\alpha_3 p^3\cdots), \tag{1}$$

of order higher than p. An admittedly incomplete analysis of the effect of phonon-phonon interactions, carried out by one of us (T.R.) in collaboration with Ponzano and Barucchi,[5] shows that in a realistic fluid one cannot exclude the presence of the linear term.[6]

Keeping this in mind we have examined the possibility of fitting the existing neutron data by Cowley and Woods[7] with a generic power expansion and of comparing these with specific-heat data. In doing this one must necessarily strike a balance by suitably adjusting the range of momenta in order to avoid the following pitfalls: (a) If the range is too narrow there are not enough data to gather meaningful information of the coefficients of the power expansion. (b) If the range is too wide it may go beyond the radius of convergence of the power-series expansion

$$\frac{\epsilon(p)}{c\hbar p}=\sum_{i=0}\alpha_i p^i. \tag{2}$$

In this case the coefficients of the fitted powers may bear no relation to the actual power expansion. This phenomenon should show up as an instability, as wild oscillations of these coefficients versus the selected range, and as a general worsening of the fit.

We fitted $\epsilon(p)/p$ and $[\epsilon(p)/p]^2$, as experimentally given in Ref. 7, with standard Forsythe polynomials[8,9] and rearranged them into conventional powers both in the variable p and in p^2. In the limit $p\to 0$ we have taken $\epsilon(p)/\hbar p=c=237$ m/sec $=(18.12$ Å °K$)k_B/\hbar$ with an error of 0.1%. (A similar analysis has been performed by taking $c=238.5\pm1.5$ m/sec with inessential changes in the final conclusion.) This was done for the range $0\leqslant p\leqslant q$ where q increased from 1.1 to 3.6 Å^{-1}. We report some of our results in Table I. The following conclusions seem to be apparent from our fits:

(1) Power expansions in p are cleanly favored over powers in p^2, thus bringing some evidence in favor of the Feenberg conclusion.

(2) Phenomenon (b) occurs in the function $\epsilon(p)/p$ for $q\approx1.9$ Å^{-1}, that is, the roton minimum. The same phenomenon occurs in $[\epsilon(p)/p]^2$ for $q\simeq2.7$ Å^{-1} and beyond. The conclusion seems that $\epsilon(p)/p$ has square-root branch points near the roton minimum and that these do not show up when $\epsilon(p)/p$ is squared. In fact our best fit for $[\epsilon(p)/p]^2$ has zeros at $p_1=2.9$, $p_2=1.9+0.3i$, $p_3=19-0.3i$, $p_4=-0.67$, and $p_5=-2.5$. This square-root behavior is expected on two counts: from the Bogoliubov[10] formula and from the fact that eigenvalues of a matrix generally have these singularities as a function of a parameter.

(3) If the fit for $\epsilon(p)/p$ and $[\epsilon(p)/p]^2$ is carried out in the variable p^2, the phenomenon (b) occurs almost immediately. Beyond 1.9 Å^{-1} the evidence in favor of fitting $[\epsilon(p)/p]^2$ in p is overwhelming.

Our best fit is then

$$\epsilon(p)=c\hbar p(1+0.55p-1.35p^2+0.26p^3+0.19p^4-0.05p^5)^{1/2}, \qquad (3)$$

where p is measured in Å^{-1}, $\epsilon(p)$ in °K, and the numerical coefficients are expressed in Å^n with $n=1,\cdots,5$.

We see that there appears a term in p in accordance with our proposal. The values of the coefficients in the power expansion seem to be fairly stable and consistent over a wide range for q. Since the same obtains for $\epsilon(p)/p$ for $0\leq q \leq 1.9$ Å^{-1} we expect $\epsilon(p)/p$ to be analytic in the circle $p=1.9$ Å^{-1}. This implies that $[\epsilon(p)/p]^2$ should be positive in the range $0\leq q\leq 1.9$ Å^{-1}.

Table I. Compound table of some of our fits of the liquid-He II dispersion curve relative to the interval $0\leq p\leq q$. The fitting polynomials are of the type

$$f(x)=\sum_{i=0}^{N}\alpha_i x^i \quad (\alpha_0=1),$$

where N can be 3, 4, or 5 and x stands for either p (first and third columns) or p^2 (second and fourth columns). The number in front of the coefficients in each case is the χ^2 of the fit.

q		ϵ/cp versus p		ϵ/cp versus p^2		$(\epsilon/cp)^2$ versus p		$(\epsilon/cp)^2$ versus p^2	
	α_1	0.46		0.21		0.99		0.34	
1.1	α_2	−1.04	1.75	−0.91	2.17	−2.37	1.67	−1.59	3.13
	α_3	0.33		0.43		0.94		0.79	
	α_1	0.43		−0.05		0.81		−0.21	
1.4	α_2	−0.98	2.28	−0.28	17.64	−1.95	3.42	−0.32	22.62
	α_3	0.30		0.09		0.71		0.12	
	α_1	0.50		−0.03		1.06		−0.21	
1.7	α_2	−1.23	2.72	−0.35	17.94	−2.80	3.04	−0.37	24.55
	α_3	+0.58		0.17		1.60		0.21	
	α_4	−0.10		−0.02		−0.29		−0.03	
	α_1	0.63		0.05		1.28		−0.11	
	α_2	−1.82		−0.57		−3.66		−0.64	
1.9	α_3	1.48	3.56	0.37	12.18	2.78	2.99	0.44	19.30
	α_4	−0.66		−0.01		−0.98		−0.12	
	α_5	0.12		~0		0.14		0.01	
	α_1	0.68		−0.07		1.17		−0.28	
	α_2	−2.02		−0.29		−3.25		−0.26	
2.1	α_3	1.76	4.20	0.15	23.41	2.29	3.53	0.17	30.62
	α_4	−0.82		−0.03		−0.73		−0.04	
	α_5	0.16		~0		0.10		~0	
	α_1	0.11		−0.16		0.77		−0.41	
	α_2	−0.02		−0.11		−2.01		−0.06	
2.4	α_3	−0.63	37.01	0.04	36.28	0.93	8.79	0.06	41.86
	α_4	0.37		~0		−0.10		−0.01	
	α_5	−0.05		~0		−0.01		~0	
	α_1	−0.25		−0.15		0.55		−0.44	
	α_2	1.14		−0.12		−1.35		−0.01	
2.6	α_3	−1.91	71	0.04	38.13	0.26	14.79	0.03	46.18
	α_4	0.95		~0		0.19		~0	
	α_5	−0.15		~0		−0.05		~0	
	α_1	−0.29		−0.12		0.57		−0.44	
	α_2	1.25		−0.15		−1.43		−0.01	
2.7	α_3	−2.03	71.76	0.06	45.15	0.34	15.05	0.03	46.18
	α_4	1.00		~0		0.15		~0	
	α_5	−0.15		~0		−0.05		~0	

As we said, the extrapolated polynomial $[\epsilon(p)/p]^2$ shows two negative real roots at −0.67 and −2.5 Å^{-1}, but these may be brought to coincide by increasing the coefficient of the fifth Forsythe polynomial well within its variance and without appreciably changing the physical region. Finally the evaluation of the specific heat at low temperature using the fit (3) shows an acceptable agreement with Ref. 1. Since errors are not quoted there, it is difficult to be more specific. In Fig. 1 we show our spectrum (3) together with the experimental data (and their errors).[7] In Fig. 2 our calculated temperature dependence of the constant-volume specific heat, again compared with the experiment,[1] is reported. Beyond 2.6 Å^{-1} our fit becomes markedly inadequate, as expected. Here, indeed, hybridization of levels occurs and this corresponds to the appearance of more branching points. Also at about 2.7 Å^{-1} the dispersion curve reaches twice the roton energy, thus becoming unstable.

The best treatment of the instability threshold is due to Pitaevskii.[11] According to Pitaevskii $\epsilon(p)$ should never go above 2Δ, but rather reach

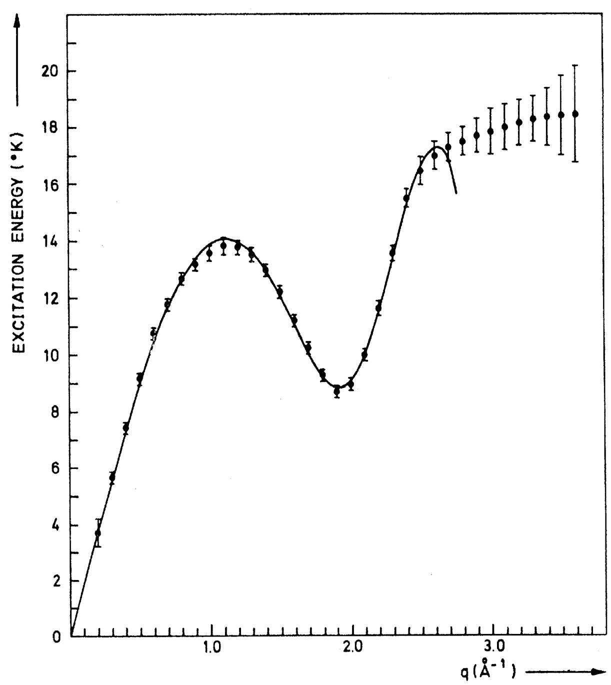

FIG. 1. The dispersion curve in liquid He II as given by our formula (3) (solid line) compared with the experimental data taken from Cowley and Woods (Ref. 7).

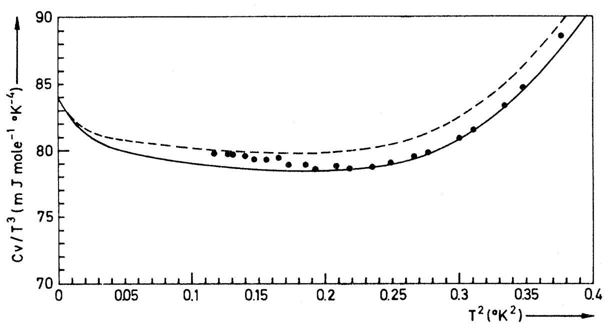

FIG. 2. Experimental constant-volume specific heat [as given in Ref. 3] compared with the theoretical result obtained using formula (3) (solid line). Dashed line represents the result obtained when formula (3) is modified (within a standard deviation) by adding a multiple of the Forsythe polynomial P_5 in such a way as to have $\epsilon^2(p) \geq 0$ for negative p.

it with an exponential behavior

$$\epsilon(p) = 2\Delta - \alpha \exp[-a/(p_c - p)], \tag{4}$$

(α and a are constants and $p_c < 2p_0$) and abruptly terminate at $p = p_c$. This follows from summing up divergent diagrams in the Dyson equation. What the argument really shows is that either the above is true or $a = 0$. In the forthcoming work of Ponzano, Barucchi, and Regge[5] some arguments are reported supporting this possibility. If this occurs, then our arguments show that $\epsilon(p)$ should cross the threshold at $p_0\sqrt{2}$ where p_0 is the roton momentum; that is, at threshold the excitation should decay into two orthogonal rotons. This relation is well satisfied for normal vapor pressure; the threshold would occur at 2.7 Å^{-1}. At higher pressure the only available curve at 25.3 atm is published[12] without quoted errors and is in rough qualitative agreement with our prescription, but as already observed it seems to reach twice the roton minimum at lower momenta than the predicted $p_0\sqrt{2} \simeq 2.9$ Å^{-1}. We think that any conclusion at the moment is premature since it is difficult to evaluate errors. Also specific-heat measurements seem to favor a slightly higher roton minimum (7.6°K instead of 7°K) which would displace the threshold in the right direction.

We do not claim here that our conclusions are in any sense final, but merely suggestive, since more refined data could well change the picture. However they show that some precise statement on the local analytic properties of $\epsilon(p)$ is probably within reach, and this is no doubt relevant to the long-sought theory of a realistic He II. Of interest, also, would be an analysis of the known discrepancies for sound attenuation with the proposed fit.

Finally, the liquid-sturcture factor as calculated from (3) with the Feynman formula $S(K) = K^2/\epsilon(K)$ has a downward-facing concavity at low momenta in qualitative agreement with the hump hypothesized by Massey.[13]

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