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Hydrodynamics of Nematic Liquid Crystals*

Huey-Wen Huang
Department of Molecular Biophysics and Biochemistry, Yale University, New Haven, Connecticut 06520
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The Ericksen-Leslie theory for nematic liquid crystals is simplified and improved. The most general linear hydrodynamic equations are obtained following the conservation laws, thermodynamics laws, and symmetry properties. The rigorous low-frequency, long-wavelength equations are then deduced by taking such a limit. It is a simple alternative way to derive the theory of Forster, Lubensky, Martin, Swift, and Pershan. It would be of interest to study the more general equations experimentally.

The hydrodynamic theory of nematic liquid crystals has been discussed by many authors.[1-5] Most recently Forster, Lubensky, Martin, Swift, and Pershan[6] (FLMSP) have derived the low-frequency, long-wavelength equations for the case of zero external field. In this note we rederive the FLMSP theory by simplifying and improving the Ericksen-Leslie theory.[3,7] We discard unnecessary assumptions such as the transformation properties between noninertial frames of reference. Following the conservation laws, thermodynamics laws, and symmetry properties, we obtain the *most general* set of linear hydrodynamic equations. The rigorous low-frequency, long-wavelength equations are then obtained by taking such a limit. Our more general equations are, however, subject to experimental test.

The macroscopic description of the state of a nematic liquid crystal is effected by means of the local density ρ, velocity $\vec{v}$, pressure p, and unit vector $\hat{n}$ denoting the orientation of the molecular axis. In this note, we shall neglect thermal expansion and conduction.

The conservation of mass is expressed by

$$d\rho/dt=-\rho\partial v_i/\partial x_i. \tag{1}$$

(A summation from 1 to 3 is implied for repeated indices.) The stress tensor σ_{ij} is defined by the conservation of translational momentum:

$$(d/dt)\int_V \rho v_i\, dV=-\oint_A p\, dA_i+\oint_A \sigma_{ij}\, dA_j, \tag{2}$$

where V denotes any material volume with surface boundary A and dA_i the vector component of a surface element. In Eq. (2) a potential force has been neglected because it is of second order in variations.[1,3]

We define the angular velocity of a molecular axis to be

$$\omega_i=\epsilon_{ijk}n_j^0\partial n_k/\partial t\simeq\epsilon_{ijk}n_j^0 dn_k/dt \tag{3}$$

with corresponding moment of inertia per unit mass I_i; $\hat{n}^0$, the equilibrium nematic direction, is taken to be the 3 axis. Under the assumption that $I_1=I_2=I$ while $I_3\sim 0$, the conservation of angular momentum in the 1 and 2 directions can be written as

$$(d/dt)\int_V(\rho\epsilon_{ijk}x_j v_k+\rho I\omega_i)dV=-\int_V\epsilon_{ijk}n_j^0(dF/dn_k)dV-\oint_A p\epsilon_{ijk}x_j\, dA_k+\oint_A\epsilon_{ijk}x_j\sigma_{kl}\, dA_l+\oint_A\Gamma_{ij}\, dA_j, \tag{4}$$

where F is the potential energy per unit volume including that due to the external field,[2,4] and the torque tensor Γ_{ij}, which is defined by this equation, balances the other torques. Whether Γ_{ij} exists or not can only be determined by experiment. Equation (4), which is second order in time, can be regarded as the single-collision-time modification[8] of the FLMSP equation for ω_i.

From Eqs. (2) and (4), one obtains the equation for rotation of the molecular axis:

$$\rho I d\omega_i/dt=-\epsilon_{ijk}n_j^0 dF/dn_k-\epsilon_{ijk}\sigma_{jk}+\partial\Gamma_{ij}/\partial x_j. \tag{5}$$

We see that the antisymmetric part of σ_{ij} arises naturally and is indispensable.

Finally we use the conservation of energy,

$$d\epsilon = v_i dv_i + I\omega_i d\omega_i + d(F/\rho) + Tds + (p/\rho^2)d\rho, \tag{6}$$

where ϵ and s are the energy and entropy per unit mass, respectively, to obtain the rate of entropy production:

$$\rho T ds/dt = v_{ij}{}^s\sigma_{ij}{}^s + (v_{ij}{}^a + \epsilon_{ijk}\omega_k)\sigma_{ij}{}^a + \omega_{ij}{}^s\Gamma_{ij}{}^s + \omega_{ij}{}^a\Gamma_{ij}{}^a, \tag{7}$$

with

$$\sigma_{ij}{}^s = \tfrac{1}{2}(\sigma_{ij}+\sigma_{ji}),\quad \sigma_{ij}{}^a = \tfrac{1}{2}(\sigma_{ij}-\sigma_{ji}),\quad v_{ij}{}^s = \tfrac{1}{2}(v_{ij}+v_{ji}),\quad v_{ij}{}^a = \tfrac{1}{2}(v_{ij}-v_{ji}),\quad v_{ij} = \partial v_i/\partial x_j, \tag{8}$$

and similar definitions for $\Gamma_{ij}{}^s$, $\Gamma_{ij}{}^a$, $\omega_{ij}{}^s$, and $\omega_{ij}{}^a$. The expression for the entropy production has to satisfy the following conditions: (1) $ds/dt \geq 0$; (2) Galilean invariance; (3) rotational symmetry along $\hat{n}^0$ direction; (4) invariance under the transformation $\hat{n} \to -\hat{n}$, $x_1 \to x_1$, $x_2 \to -x_2$, $x_3 \to -x_3$; (5) invariance under the transformation $x_1 \to -x_1$, $x_2 \to x_2$, $x_3 \to x_3$; and (6), since the internal friction occurs only when there is a relative motion between various parts of the system, the viscosity (dissipation) should vanish if the entire system undergoes a uniform translation or rotation. The most general form for the entropy production satisfying these conditions is

$$\begin{aligned} T\rho ds/dt = {} & 2\nu_2(v_{ij}{}^s)^2 + 4(\nu_3-\nu_2)(v_{ik}{}^s n_k{}^0)^2 + 2(\nu_1+\nu_2-2\nu_3)(n_k{}^0 n_l{}^0 v_{kl}{}^s)^2 + (\nu_4-\nu_2)(v_{kk}{}^s)^2 \\ & + 2(\nu_5-\nu_4+\nu_2)v_{ii}{}^s n_k{}^0 n_l{}^0 v_{kl}{}^s + \gamma\{(\delta_{ik}-n_i{}^0 n_k{}^0)n_j{}^0 n_l{}^0[\lambda v_{kl}{}^s + (v_{kl}{}^a + \epsilon_{klm}\omega_m)]\}^2 \\ & + \zeta_1[(\delta_{kl}-n_k{}^0 n_l{}^0)\omega_{kl}{}^s]^2 + \zeta_2[(\delta_{ik}-n_i{}^0 n_k{}^0)(\delta_{jl}-n_j{}^0 n_l{}^0)\omega_{kl}{}^s]^2 + \zeta_3[(\delta_{ik}-n_i{}^0 n_k{}^0)n_j{}^0 n_l{}^0\omega_{kl}]^2 \\ & + \zeta_4[(\delta_{ik}-n_i{}^0 n_k{}^0)(\delta_{jl}-n_j{}^0 n_l{}^0)\omega_{kl}{}^a]^2. \end{aligned} \tag{9}$$

We have used the coefficients of FLMSP as closely as possible ($\gamma = \gamma_1$ of FLMSP). The terms with coefficients ζ as well as Γ_{ij} are first introduced here. The cross terms between v_{ij} and ω_{ij} are forbidden by the conditions (4) and (5). The positivity of the entropy production implies

$$\begin{aligned} & \nu_2 \geq 0,\quad \nu_4 \geq 0,\quad 2(\nu_1+\nu_5) \geq \nu_4-\nu_2,\quad \nu_4(2\nu_1+\nu_2) \geq (\nu_4-\nu_5)^2, \\ & \nu_3 \geq 0,\quad \gamma \geq 0,\quad \zeta_2 \geq 0,\quad 2\zeta_1+\zeta_2 \geq 0,\quad \zeta_3 \geq 0,\quad \zeta_4 \geq 0. \end{aligned} \tag{10}$$

$\sigma_{ij}{}^s$, $\sigma_{ij}{}^a$, and Γ_{ij} are obtained by taking half of the derivative of the expression (9) with respect to $v_{ij}{}^s$, $(v_{ij}{}^a + \epsilon_{ijk}\omega_k)$, and ω_{ij}, respectively. They are purely dissipative as can be seen by examining the time reversal of Eqs. (2) and (5). Onsager's reciprocal relations[9] for the coefficients are automatically satisfied.

If we now take the low-frequency, long-wavelength limit, the terms $\rho I d\omega_i/dt$ and $\partial\Gamma_{ij}/\partial x_j$ drop out because they are of higher order in frequency and wave vector. Equations (2) and (5) reduce to the FLMSP equations.

The existing data have been compared and found to agree with the FLMSP[6] theory. To use Eq. (5) one has to go to a higher frequency and a larger wave vector. There is one question left unanswered: Should one also consider the single-collision-time modification of Eq. (2)? Only more extensive experimental measurements can answer this question. We have used the conservation equations of translational and rotational momenta because their mutual relation and their association with Newton's second law greatly simplify the derivation, whereas such a physical picture is less obvious in FLMSP.

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Modulated Electron-Beam Excitation of Low-Frequency Plasma-Cavity Modes*

V. P. Bhatnagar and W. D. Getty
Electron Physics Laboratory, Department of Electrical Engineering, The University of Michigan, Ann Arbor, Michigan 48104
(Received 5 March 1971)

Ion heating and relatively large radial, rf electric fields have been observed in an experiment in which an electron beam modulated near the ion plasma frequency excites the resonances of a partially filled plasma cavity. Predictions based on a theoretical self-consistent model for a finite-length beam-plasma system are in good agreement with the experimental observations.

Several reports have been published on observations of plasma ion heating by a modulated electron beam when the frequency of modulation is near the ion plasma frequency. In this paper measurements and calculations are presented which show that the modulated electron beam excites resonances of a partially filled plasma cavity formed by the metal vacuum envelope, and that at the cavity resonant frequencies a relatively large radial, rf electric field is excited in the plasma which produces the observed ion heating. This method of electron-beam excitation of ion oscillations is fundamentally different from other methods[1,2] since it does not require the presence of a strong beam-plasma instability in the frequency range in which rf electric fields are desired. The excitation frequency is determined by the plasma density and geometry, and can be at low frequencies where ion oscillations can be excited. The electron-ion lower hybrid resonance frequency $\omega_{lh}^{-2}=(\omega_{ce}\omega_{ci})^{-1}+(\omega_{pi}^2+\omega_{ci}^2)^{-1}$ is of particular interest because at this frequency the ions oscillate with an average kinetic energy equal to or greater than that of the electrons in the presence of an rf electric field.

The main characteristics of modulated-beam experiments with a beam-generated plasma have been reported by Haas and Dandl[3] for modulation frequencies near ω_{ci} and ω_{pi}. Beeth, Haas, and Eisner[4] have reported measurements on ion heating in nonbeam-generated plasma at frequencies above the ion plasma frequeny. Beam-excited oscillations have been observed when the modulation frequency is near the ion-ion hybrid resonance frequency in a three-component plasma.[5] In work reported earlier, the present authors[6] observed ion heating in a hydrogen plasma at one to three different frequencies immediately above ω_{pi}. In all of these experiments the beam current is modulated by applying an rf modulating voltage to a current-controlling electrode in the electron gun.

The plasma is generated by the electron beam in a metal vacuum system 10 cm in diameter and 61 cm in length measured from the electron gun to the grounded beam collector. The electron beam is produced by a triode electron gun with dc accelerating potentials of 400 to 1000 V. The grid is biased negatively to adjust the average beam current and to insure that sinusoidal current modulation is obtained. Either hydrogen, deuterium, neon, or argon is admitted continuously at a rate that produces pressures in the range of 10^{-4} to 10^{-3} Torr. Beam current and gas pressure are chosen to obtain a beam-generated plasma in the "quiescent" mode[7] with densities of 5×10^8 to 5×10^9 cm^{-3} and a typical electron temperature of 6 eV. The ratio of plasma density to beam density is approximately 25, and the plasma diameter is approximately 6 mm. An axial magnetic field of 300 to 400 G is used. There is a weak magnetic mirror at each end of the system.

The main diagnostic methods used are Langmuir probes with coaxial leads and a gridded retarding-potential probe. Two movable Langmuir probes are used to measure the rf electric field as a function of radius and axial position, and a third probe is used to measure plasma density. The rf probe signal is fed to a matched 100×