and experimental single-particle energies of ^{16}O and ^{40}Ca. Other approaches have also been proposed[15,16] which reproduce the shift between the $T=0$ and $T=1$ states in doubly closed-shell nuclei, resulting from empirical observations in parametrizing effective matrix elements. However, none of these[14-16] provides an adequate explanation for the mechanism leading to the shift so as to lead to a satisfying framework for the study of configuration mixing. In addition, our model has the advantage over the modified surface delta interaction[16] of permitting the use of either realistic or effective interactions. As well as the study of the improvements already described, the changes in the description of the collective states brought in by this model will be interesting. A complete RPA calculation with these shifts included has therefore been undertaken.[12]

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[1]T. P. Elliott and B. H. Flowers, Proc. Roy. Soc., Ser. A 242, 57 (1957); G. E. Brown, L. Castillejo, and J. A. Evans, Nucl. Phys. 22, 1 (1960).

[2]V. Gillet and N. Vinh Mau, Nucl. Phys. 54, 321 (1964).

[3]H. A. Mavromatis, W. Markiewicz, and A. M. Green, Nucl. Phys. A90, 101 (1967).

[4]V. Gillet and E. A. Sanderson, Nucl. Phys. A91, 292 (1967).

[5]A. E. L. Dieperink, H. P. Leenhouts, and P. J. Brussaard, Nucl. Phys. A116, 556 (1968).

[6]A. Jaffrin and G. Ripka, Nucl. Phys. A119, 529 (1968).

[7]J. Blomqvist and T. T. S. Kuo, Phys. Lett. 29B, 544 (1969).

[8]M. Weigel, L. Garside, and P. K. Haug, UCRL Report No. UCRL-19940 (to be published).

[9]P. M. Endt and C. Van Der Leun, Nucl. Phys. A105, 1 (1967); R. J. Philpott and P. P. Szydlik, Phys. Rev. 153, 1039 (1967).

[10]R. Trilling, unpublished.

[11]J. Blomqvist, Phys. Lett. 33B, 541 (1970).

[12]F. Petrovich, R. Schaeffer, and R. Trilling, to be published.

[13]A. E. L. Dieperink, Nucl. Phys. A160, 171 (1971).

[14]A. Bohr and B. Mottelson, *Nuclear Structure* (Benjamin, New York, 1969), Vol. 1, p. 329.

[15]R. K. Bansal and J. B. French, Phys. Lett. 11, 145 (1964); L. Zamick, Phys. Lett. 19, 580 (1965).

[16]P. W. M. Glaudemans, P. J. Brussaard, and B. H. Wildenthal, Nucl. Phys. A102, 593 (1967).

Removal of Ambiguities in an S-Matrix Analysis*

P. L. Jolivette†
University of Wisconsin, Madison, Wisconsin 53706
(Received 16 April 1971)

In general, an S-matrix or phase-shift analysis yields several ambiguous solutions that give identical fits to the data. Computer studies of a simple system involving only a few resonances suggest that the unphysical solutions show characteristic correlations (pseudoresonances) in different partial waves near a true resonance in one partial wave. Therefore, the physical solution requires the fewest resonant states. We apply this criterion to an analysis of $^{16}O(d,\alpha_1)^{14}N$.

Even for purely elastic scattering a phase-shift analysis using only differential cross sections at one energy will have several solutions all of which give identically the same cross sections at all angles. An example is the Minami ambiguity[1] for elastic scattering of spin-$\frac{1}{2}$ by spin-0 particles where the interchange of all pairs of phase shifts with the same J but different $l=J\pm\frac{1}{2}$ leaves the angular distribution unaltered. Several authors have discussed the multiplicity of such ambiguities for cases of elastic scattering[2-4] and reactions.[5] Gersten[4] has, in addition, given a prescription for finding the entire set of solutions given one of them.

The selection of the correct solution from the ambiguous set is possible in certain cases. For elastic scattering of charged particles the long-range Coulomb interaction defines the phase shifts for high l. Because these would be altered in the transformation to other solutions, the ambiguities cannot exist. Polarization measurements for purely elastic spin-$\frac{1}{2}$–spin-0 scatter-

ing reduce the ambiguity from 2^n, where n is the number of phase shifts used, to 2 if no reaction channels are open.[2] In some cases nonunitarity of the S matrix may exclude certain solutions (e.g., see Gersten,[4] p. 540). If an energy-dependent analysis is done, then the behavior of the phase shifts at low energies and arguments based on continuity and conformance to specific models might help. In addition, elastic phase shifts must obey Wigner's theorem, $d\delta/dk \geq -a$, where a is the cut-off radius for the interaction.[6] However, none of these approaches were of help in recent S-matrix analyses of isospin-forbidden reactions undertaken at Wisconsin.[5,7,8] These reactions have in common a very simple spin and parity combination ($0^+ + 1^+ \rightarrow 0^+ + 0^+$) and very low cross sections. The special combination of spins and parities makes the analysis straightforward,[5] but gives an energy-independent analyzing power so that use of polarized deuterons (1^+) adds no information.[9] However, the degree of ambiguity is small: 2^L where partial waves 1 through L are needed to fit the data ($l=0$ is forbidden). There are also two trivial transformations, reflections about the real and imaginary axis (or a reflection and an arbitrary rotation), which preserve the $|S_L|$, the magnitudes of the S-matrix elements for the individual partial waves. In addition, the magnitude of the highest partial wave used in fitting the data, $|S_L|$, is identical for all 2^L solutions. Thus there are 2^{L-2} sets of solutions with different values of the $|S_l|$. For the reactions we studied, $^{16}O(d, \alpha_1)^{14}N$ and $^{14}N(\alpha, \alpha_1)^{14}N$, the $|S_l|$ of all solutions are always small, <0.25, so unitarity requirements are of no help. When an additional partial wave is required as the energy increases, it enters with infinitesimal magnitude so that each previous solution splits smoothly into two solutions. Any level in the Lth partial wave is of course unambiguously identified because for the Lth partial wave all solutions are identical. Even for $l < L$ a strong resonance often appears in all solutions and so is unambiguous.

An example of the multiple solutions is shown in Fig. 1. In most of the region $L=3$ or 4. If $L=3$, there are only two solutions (2^{L-2}) and below $E_d = 5.5$ MeV the magnitude of S_4 is small enough that the additional splitting is minimal. The two solutions (one primed, the other unprimed) are shown for $l=1$ and $l=2$ ($l=3=L$ is the same for both).

Often it is difficult to guarantee that one stays with the same solution as the energy changes.

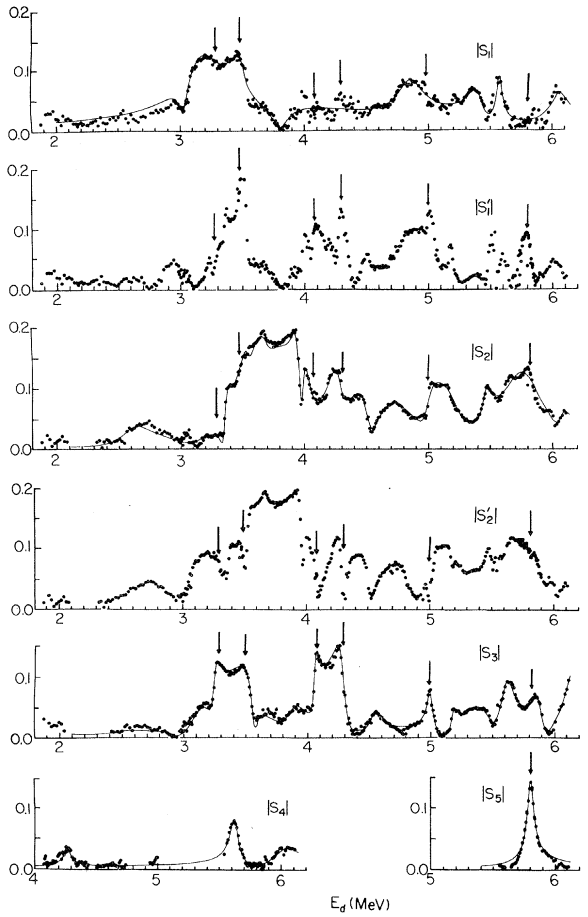

FIG. 1. The dots are the $|S|$-matrix elements from an analysis of $^{16}O(d,\alpha_1)^{14}N$ differential cross sections (Ref 8). The scattering amplitude is given by Eq. (1). When partial waves 1 through L are used, there are 2^{L-2} sets of $|S_l|$ that reproduce the cross sections identically. When $L=3$ there are two solutions and these have the same $|S_3|$. The two solutions correspond to the primed and unprimed $|S_l|$. The unprimed solution is the "physical" one according to a study of the curves at the energies indicated by the arrows. Here the invariant Lth partial wave has strong structure. In the "physical" solution $l=1$ or $l=2$ are relatively smooth at these energies while the primed solution shows a resonance in either $l=1$ or $l=2$ and a corresponding dip in the other so as to conserve total cross section. The line through the "physical" solution is a fit to the S-matrix elements using

$$S_l = \sum_\lambda r_\lambda \exp(i\beta_\lambda)/(E - E_\lambda + i\tfrac{1}{2}\Gamma_\lambda)$$

(Ref. 5).

The problem is worst when two solutions are very similar or are changing rapidly. For example, the two solutions in Fig. 1 are not qualitatively so different and both represent a region of many overlapping levels in the compound nucleus

with no detectable background contribution, and no direct reaction component. In addition, the solutions are identical in all respects at frequent intervals, several points per MeV.

In such situations continuity is best checked by the same method used to calculate all the solutions.[4] For our particular case[5] we write the scattering amplitude $f(\theta)$ (which is a power series in $\cos\theta$) in product form:

$$f(\theta)=\frac{\lambda}{2\sqrt{3}}\sum_{l=1}^{L}\frac{2l+1}{[l(l+1)]^{1/2}}S_l\frac{dP_l\cos\theta)}{d\theta} \quad (1a)$$

$$=(1-\cos^2\theta)^{1/2}\sum_{n=0}^{L-1}a_n\cos^n\theta \quad (1b)$$

$$=(1-\cos^2\theta)^{1/2}c\prod_{m=1}^{L-1}(\cos\theta-b_m). \quad (1c)$$

The b_m are the complex roots of the scattering amplitude, and the a_n are linear combinations of the complex S-matrix elements. The S_l are then easily calculated from the b's. The various solutions are found by complex conjugation of some combination of the b's. This procedure merely rearranges the double product for the cross section $d\sigma/d\Omega=f^*(\theta)f(\theta)$, and thus leaves it unchanged.

To learn the energy dependence of the solutions one then sorts the b's and determines the signs of their real parts. Difficulties arise only in deciding whether $\mathrm{Im}(b_m)$ changes sign when it approaches zero. In general, one must rely on the trend of $\mathrm{Im}(b_m)$, noting that it changes rapidly only if the cross section changes quickly. Thus, for example, if $\mathrm{Im}(b_m)$ approached zero at a high angle, it was assumed that it changed sign, but if the angles of approach were low and $\mathrm{Im}(b_m)$ remained small for a reasonable distance, the sign was not changed. The general rule avoids unnatural appearing kinks in the trajectory. The solutions will form pairs whenever b_m is real, and if this happens when $-1<b_m<1$, the cross section is zero at $\theta=\cos^{-1}b_m$. This procedure is not foolproof and the solutions presented here were corrected *ex post facto* to provide a continuous solution that conforms to the "physical" criterion below.

With continuous solutions assumed, we still lack clear physical criteria for choosing among ambiguous solution sets. Instead, we use Occam's razor to select the "physical" solution as the one implying the fewest nuclear states. As will be seen, this corresponds to the requirement that a strong level in one partial wave is not closely correlated with structure in several other partial waves.

To gain insight into appropriate selection criteria we performed the following computer experiment: Using the simple $(0^+ +1^+ \rightarrow 0^+ +0^+)$ system, we postulate a "physical" solution which has only two Breit-Wigner resonances, one in the $l=3$ partial wave and the other in the $l=2$ partial wave but at a slightly lower energy. The correct S-matrix elements are then defined as

$$S_1=0,\quad S_2=\frac{r_2\exp(i\beta_2)}{E-E_2+\frac{1}{2}i\Gamma_2},$$

$$S_3=\frac{r_3\exp(i\beta_3)}{E-E_3+\frac{1}{2}i\Gamma_3},\quad S_{4,5,\ldots}=0. \quad (2)$$

Column (*a*) of Fig. 2 shows the absolute values of these matrix elements for our assigned parameters $r_1=r_2$, $\Gamma_1=\Gamma_2$, $E_3=E_2-\Gamma$. This choice defines the $|S_l|$ and the *total* cross section for the "physical" solution. The *differential* cross section remains undetermined until we fix the phases β_l (or more accurately the phase difference $\varphi=\beta_2-\beta_3$ since cross sections are independent of the absolute phase). The three choices of $\varphi-0$, $\pi/4$, and $\pi-$result in quite different behavior of the differential cross sections.

With φ chosen, we next use Gersten's prescription[4] to generate all the other ambiguous solutions for that φ. In our simple case there

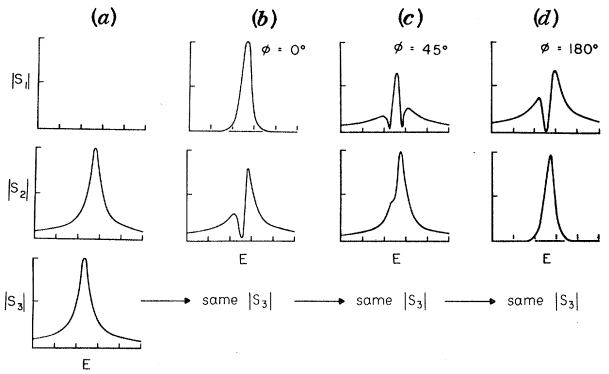

FIG. 2. Column (*a*) shows $|S_l|$ for a "physical" solution (correct by definition) consisting of a Breit-Wigner resonance in $l=2$ and another in $l=3$. The S-matrix elements are $S_1=0$, $S_2=re^{i\varphi}/(E-E_2+\frac{1}{2}i\Gamma)$, and $S_3=r/(E-E_3+\frac{1}{2}i\Gamma)$. As $L=3$ there are only two solutions and they have the same $|S_3|$. The $|S_l|$ for the "physical" solution are independent of φ. The secondary solution for each of three values of φ is shown in columns (*b*)-(*d*). These secondary solutions are more complicated than the "physical" solution and require many more resonant states for their explanation. The extra structure occurs at or very near the energies of the resonances in the "physical" solution. An increase in $l=1$ must correspond to a decrease in $l=2$ relative to the "physical" solution in order to conserve total cross section.

is only one other solution since $L=3$ and $2^{L-2}=2$. The absolute value of this alternate (and by definition unphysical) solution is shown in column (b) for $\varphi=0$, column (c) for $\varphi=\frac{1}{4}\pi$, and column (d) for $\varphi=\pi$. $|S_3|$ is identical for all solutions since $L=3$. These secondary unphysical solutions are all continuous and would be indistinguishable from the physical solution in each case for any experiment performed on the single reaction channel represented.

The common characteristic of all the unphysical solutions is the introduction of great complexity in all partial waves $l<L$. In particular, pseudoresonances appear in the neighborhood of the true resonances and there are obvious correlations in energy of these pseudoresonances with the true resonances. Since the $|S_l|$ for our isospin-forbidden reaction should on physical grounds be a coherent sum of Breit-Wigner amplitudes, the simple appearances of $|S_1|$ for $\varphi=0$ and $|S_2|$ for $\varphi=\pi$ are misleading because many overlapping states are required to eliminate the long Breit-Wigner tails characteristic of a single resonance.

The computer experiment teaches us then to favor the solution requiring the fewest resonant states and in particular alerts us to question solutions having extra dips or peaks in low partial waves at or near the resonant energy of higher partial waves (in other words, Occam's razor).

Figure 1 shows how we apply this criterion to the reaction $^{16}O(d, \alpha_1)^{14}N$. The arrows indicate regions where the invariant Lth partial wave shows strong narrow structure. Near these energies the primed solution shows peaks or dips while the unprimed solution is smoother. Inspection shows that a peak in one of the partial waves, $l=1$ or $l=2$, for the primed solution corresponds to a dip in the other. This correlation results from a transformation on the smoother "physical" solution which conserves total cross section. Thus we conclude that the unprimed "physical" solution is more likely to be the "real" solution.

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†Present address: University of Notre Dame, Notre Dame, Ind. 46556.

[1]S. Minami, Progr. Theor. Phys. 11, 213 (1954).
[2]N. P. Klepikov, Zh. Eksp. Teor. Fiz. 41, 1187 (1961) [Sov. Phys. JETP 14, 846 (1962)].
[3]R. Von Wageningen, Ann. Phys. (New York) 31, 148 (1965).
[4]A. Gersten, Nucl. Phys. B12, 537 (1969).
[5]P. L. Jolivette and H. T. Richards, Phys. Rev. 188, 1660 (1969).
[6]E. P. Wigner, Phys. Rev. 98, 145 (1955).
[7]P. B. Tollefsrud and P. L. Jolivette, Phys. Rev. C 1, 398 (1970).
[8]P. L. Jolivette, thesis, University of Wisconsin, 1971 (University Microfilms, Ann Arbor, Mich.).
[9]B. A. Jacobsohn and R. M. Ryndin, Nucl. Phys. 24, 505 (1961).

Deformation in the Second Half of the *s-d* Shell

I. Kelson
Department of Physics and Astronomy, Tel-Aviv University, Tel-Aviv, Israel, and Cyclotron Laboratory, Michigan State University, East Lansing, Michigan 48823*
(Received 5 January 1971)

A simple approximate procedure for perturbing the self-consistent Hartree-Fock equations in an enlarged single-particle space is given. Using phenomenological interactions and observed single-particle splittings, the second half of the *s-d* shell is shown to require pear-shaped intrinsic states, involving excitations from the *s-d* to the *p-f* shell.

While the existence of deformed (prolate) nuclei in the first half of the *s-d* shell has been established experimentally and theoretically,[1] the situation in the second half is not at all clear. Among the open questions is the exact nature of the "intrinsic" states, which provide an effective way of configuration mixing through projection, and in particular the possible admixture of *p-f* orbitals into them. Pear-shaped intrinsic states were used[2] for the description of the excited, collective states of O^{16} but not in an entirely self-consistent way. Such self-consistent calculations, in the framework of Hartree-Fock (HF) theory, were performed[3] in this region with special emphasis on the established[4] role of the tensor force in the two-body interaction. Using re-