quantity,[2,7] the present results open up the possibility of using data of $Y_m^{(\nu)}$ vs β to invert Eqs. (3) and (10) to attempt an explicit determination of the general functional form $f(x)$. For $m=1$ (the usual dipolar case), given $Y_1^{(\nu)}(\beta)$, one would then have to "simply" invert the equation

$$Y_1^{(\nu)}(\beta)=\frac{\int_{-1}^{1}dx\,x(1-x^2)^{(\nu-3)/2}e^{\beta f(x)}}{\int_{-1}^{1}dx(1-x^2)^{(\nu-3)/2}e^{\beta f(x)}} \tag{11}$$

to get $f(x)$.

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$$\lim_{\alpha\to 0}\alpha^{-1}C_n^{(\alpha)}(x)=2n^{-1}T_n(x).$$

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Suggestion for Choosing the Single-Particle Energies in Doubly Closed-Shell Nuclei*

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The single-particle energies in a nucleus of $A\pm1$ particles as taken from experiment are seen to be shifted when used for the calculation of the A-particle–system excited states. This shift can be described as a change in the particle-hole gap arising from an isospin-isospin interaction of the excited nucleon with the remaining core of $A-1$ particles. It lowers the $T=0$ states of ^{40}Ca by about 1.5 MeV and of ^{16}O by 3.75 MeV, and it raises the $T=1$ states one third this amount. This shift gives an account of discrepancies which are observed in all existing Tamm-Dancoff or random-phase approximation calculations of doubly closed shell nuclei.

A large number of calculations[1-8] have already been performed on doubly closed-shell nuclei. The spirit of all these works is essentially the same. One chooses a single-particle basis and splits the Hamiltonian H of the system into two parts: H_0, which is already diagonal for the particle-hole excitations $|j'j^{-1}\rangle$, and a residual interaction V. The eigenvalues of H_0 are generally taken from the experimental data on the neighboring $A\pm1$ nuclei. The matrix elements $\langle(j'j^{-1})JT|\times V|(j'j^{-1})JT\rangle$ have a component diagonal in the projection quantum numbers of the particle and the hole (and thus independent of J). This component is a correction to the particle-hole energy $\epsilon_{j'}-\epsilon_j$:

$$\langle j'm'jm|V|j'm'jm\rangle, \tag{1}$$

schematized by the graph of Fig. 1(a). In principle, it takes into account the change in the single-particle energy of a nucleon which interacts

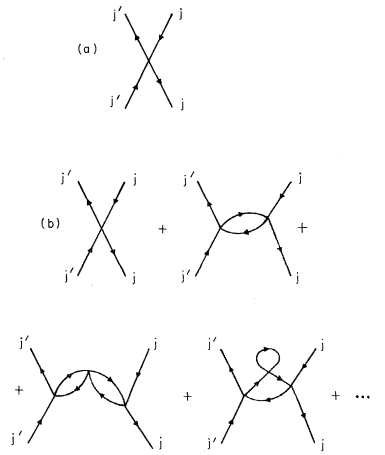

FIG. 1. (a) The correction to the single-particle energies (taken from the $A\pm1$ nuclei) usually included in a TDA or RPA calculation. (b) This same correction as a sum, calculated to all orders in the nucleon-nucleon interaction.

Table I. Shifts in the particle-hole gap arising from the isospin-isospin coupling of the excited particle with the remaining core. Except for a core with isospin $T_0 = 0$, the state $|(j'j^{-1})J, T=1\rangle$ does not have good total isospin and its shift should be considered as the weighted average of the shifts for the components $|T_0\rangle$ and $|T_0+1\rangle$ of good isospin.

State	Shift
$\|(j'j^{-1})J, T=0\rangle$[a]	$-3\epsilon_1/A$
$\|(j'j^{-1})J, T=1\rangle$[a]	ϵ_1/A
$\|(j'j^{-1})J, \tau=\pm\frac{1}{2}\rangle$[a]	$-\epsilon_1/A$

[a]Defined in the text.

with a core of the $A-1$ nucleus instead of with the A. In the same way, however, that one gets better results by using experimental single-particle energies rather than Hartree-Fock energies, the contribution of Fig. 1(a) should be corrected for higher-order processes and replaced by the sum of graphs schematized in Fig. 1(b). Except for the first-order contribution, these graphs are not included either in a Tamm-Dancoff approximation (TDA) or in a random phase approximation (RPA) calculation. (In Ref. 7 the second order was also included, as discussed below.)

The potential corresponding to the measured particle energies in the $A+1$ nucleus can be written as

$$V_0(r) + 4(\vec{t}_p \cdot \vec{T}_A/A) V_1(r). \tag{2}$$

In order to take the whole sum of graphs of Fig. 1(b) into account, we propose the replacement of (2) by

$$V_0(r) + 4(\vec{t}_p \cdot \vec{T}_{A-1}/A) V_1(r).$$

Let us neglect slight changes in the shape or the magnitude of $V_0(r)$ and $V_1(r)$. We assume in addition that the shift in the single-particle energies is proportional to the change in the potential. These approximations are not necessary for realistic calculations, but they are probably quite good. We get, therefore,

$$\delta\epsilon \sim 4(\epsilon_1/A)\{\langle A|\vec{t}_p \cdot \vec{T}_{A-1}|A\rangle - \langle A+1|\vec{t}_p \cdot \vec{T}_A|A+1\rangle\}. \tag{3}$$

$|A+1\rangle$ and $|A\rangle$ are generally taken as[1-8]

$$|A+1\rangle = a_{j'm'}{}^{\dagger}{}_{\frac{1}{2}\tau'}|T_0T_0\rangle, \tag{4}$$

$$|A\rangle = |(j'j^{-1})JTT_z\rangle = \sum_{m\tau,m'\tau'} \langle \tfrac{1}{2}\tau'\tfrac{1}{2}-\tau|TT_z\rangle\langle j'm'j-m|JM\rangle a_{j'm'(1/2)\tau'}{}^{\dagger}(-)^{j-m+1/2-\tau} a_{jm(1/2)\tau}|T_0T_0\rangle, \tag{5}$$

or, when the Pauli principle forbids the construction of a particle-hole pair of good isospin,

$$|A\rangle = |(j'j^{-1})J\tau\rangle = \sum_{m,m'} \langle j'm'j-m|JM\rangle a_{j'm'(1/2)\tau}{}^{\dagger}(-)^{j-m} a_{jm(1/2)-\tau}|T_0T_0\rangle. \tag{6}$$

The notations are standard ones[6] (T_0 is the isospin of the A-particle ground state).

In Table I, the shifts $\delta\epsilon$ are given in units of ϵ_1/A for doubly closed-shell nuclei. The value of ϵ_1 can be obtained from the symmetry energy of a particle in the $2p_{3/2}$ orbital of ^{48}Ca,

$$E_s = \epsilon_n + \Delta - \epsilon_p = 4T_0\epsilon_1/A \tag{7}$$

(ϵ_n and ϵ_p are the single-particle energies[6] and Δ the Coulomb energy). This gives $\epsilon_1 \sim 20$ MeV, that is, for the $T=0$ states, $\delta\epsilon = -1.5$ MeV in ^{40}Ca and $\delta\epsilon = -3.75$ MeV in ^{16}O. We could have equivalently considered the graphs of Fig. 1(b) as producing a shift in the hole energies which leads to the same results.

When the residual interaction is taken into account, one has to correct for the contribution (1) which is included both in $\delta\epsilon$ and in the particle-hole matrix elements. This correction is

$$\bar{V}_{j'j}{}^{T} = \sum_J \frac{2J+1}{(2j'+1)(2j+1)} \langle (j'j^{-1})JT|V|(j'j^{-1})JT\rangle. \tag{8}$$

The proper way to include it is to subtract $\bar{V}_{j'j}{}^{T}$ from the matrix element $\langle (j'j^{-1})JT|V|(j'j^{-1})JT\rangle$ before using this latter in a configuration-mixing calculation.

A well-known discrepancy of the TDA and RPA calculations (as discussed in Refs. 3-5) can be explained by our model: The excitation energies of the $T=0$ unnatural-parity states are generally predicted several MeV too high in ^{16}O and ^{40}Ca. This is seen in Table II where the results of various calculations for some $T=1$ states and $T=0$ unnatural-parity states in ^{16}O and ^{40}Ca are compared with experiments. Since these states are quite accurately described[3-5] by a single-particle excitation, the

Table II. Typical results (Refs. 1-8) obtained for $T=1$ and unnatural-parity $T=0$ states of doubly closed-shell nuclei are compared with the shift in the particle-hole gap we propose to include. The corrected values of $E_{th}-E_{exp}$ are obtained by adding the corresponding values of $E_{th}-E_{exp}$ and $\delta\epsilon-\bar{V}_{j'j}^{T}$ shown in the table. The energies are given in MeV.

^{16}O		GV[a]		MMG[a,b]	WGH[a]	
J^{π},T	$E_{exp.}$[c]	$E_{th}-E_{exp.}$	$\delta\epsilon-\bar{V}_{j'j}^{T}$	$E_{th}-E_{exp.}$	$E_{th}-E_{exp.}$	$\delta\epsilon-\bar{V}_{j'j}^{T}$[d]
$2^-,0$	8.88	1.6	-1.8	2.7	3.1	-3.4
$2^-,0$	12.52	2.1	-1.5	3.6	3.1	-3.4
$2^-,0$	13.97	2.6	-1.7	3.4	3.7	-3.4
$2^-,1$	12.96	-0	-0.3	0.4	0.3	-0.7
$2^-,1$	18.1	-0.5	-0.6	-1.2	0.4	-0.7
$2^-,1$	19.1	-0	-0.4	-0.1	0.3	-0.7
^{40}Ca		GS[a]		DBL[9]	WGH[a]	
J^{π},T	$E_{exp.}$[c]	$E_{th}-Exp.$	$\delta\epsilon-\bar{V}_{j'j}^{T}$	$E_{th}-E_{exp.}$	$E_{th}-E_{exp.}$	$\delta\epsilon-\bar{V}_{j'j}^{T}$[c]
$2^-,0$	6.02	1.3	-1.1	1.6	1.9	-1.6
$4^-,0$	5.61	1.1	-1.1	1.6	1.7	-1.6
$2^-,1$	8.47	0.2	-0.4	0.6	0.4	-0.6
$3^-,1$	7.69	0.1	-0.4	0.2	0.3	-0.6
$4^-,1$	7.66	-0.1	-0.4	0.3	0	-0.6
$5^-,1$	8.54	-0.5	-0.4	0.2	0.1	-0.6

[a]Refs 2-8. The initials refer to the various authors.
[b]Calculations referred to as *HJA* by these authors.
[c]Ref. 9.
[d]Typical values of $\bar{V}_{j'j}^{T}$ for realistic forces were used to estimate these shifts (Ref. 10). These are $\bar{V}_{j'j}^{0}\sim-0.35$ and $\bar{V}_{j'j}^{1}\sim1.95$ for ^{16}O, $\bar{V}_{j'j}^{0}\sim0.1$ and $\bar{V}_{j'j}^{1}\sim1.1$ for ^{40}Ca.

shift $\delta\epsilon$ and the correction (8) can be approximately included by adding the amount $\delta\epsilon-\bar{V}_{j'j}^{T}$ to the calculated excitation energies. These corrections are shown at the left in Table II for the calculations of Refs. 2 and 4, where a phenomenological interaction is used. Realistic interactions were used in the calculations leading to the results shown at the right in Table II. Estimates of $\bar{V}_{j'j}^{T}$ for such forces were used in arriving at the corresponding corrections shown at the extreme right in Table II.

The corrections are seen to give a reasonalbe account of the discrepancies occuring in the original[1-8] calculations for ^{16}O and ^{40}Ca, especially for the $T=0$ unnatural-parity states for which the shift $\delta\epsilon-\bar{V}_{j'j}^{T}$ cancels to a large extent the difference between theoretical[1-8] and experimental[9] excitation energies. There is, however, a tendency for the corrected results to underestimate slightly the excitation energy of the $T=1$ states (but a calculation including configuration mixing has to be done before any definitive conclusion can be drawn on this latter point). Further, in ^{208}Pb an unexplained shift of -30 to -90 keV has been noticed by Blomqvist[11] for both proton and neutron particle-hole centroid energies. This compares very favorably with our prediction (Table I) that $\delta\epsilon=-\epsilon_1/A\sim-96$ keV. The two main features of our shift—an A dependence, but no J dependence—are therefore seen to be consistent with the experimental data.

Preliminary calculations[12] show that this shift also explains why the usual[1-8] microscopic calculations overestimate, as discussed by Gillet and Sanderson,[4] the Coulomb mixing of $T=0$ and $T=1$ states in ^{40}Ca.

Blomqvist and Kuo[7] have included the second-order diagram from the sum of Fig. 1(b) in their calculation. The contribution they call A_{ph} contains, however, nondiagonal terms as well as diagonal ones, and their calculation is not directly comparable to ours. Clearly, however, their work shows that higher order contributions are needed to cancel somewhat the drastic features they obtain by including the second-order diagrams only (this is also discussed in Dieperink[13]).

A calculation very similar to (3) has been made in Bohr and Mottelson[14] for comparing calculated

and experimental single-particle energies of ^{16}O and ^{40}Ca. Other approaches have also been proposed[15,16] which reproduce the shift between the $T=0$ and $T=1$ states in doubly closed-shell nuclei, resulting from empirical observations in parametrizing effective matrix elements. However, none of these[14-16] provides an adequate explanation for the mechanism leading to the shift so as to lead to a satisfying framework for the study of configuration mixing. In addition, our model has the advantage over the modified surface delta interaction[16] of permitting the use of either realistic or effective interactions. As well as the study of the improvements already described, the changes in the description of the collective states brought in by this model will be interesting. A complete RPA calculation with these shifts included has therefore been undertaken.[12]

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Removal of Ambiguities in an S-Matrix Analysis*

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In general, an S-matrix or phase-shift analysis yields several ambiguous solutions that give identical fits to the data. Computer studies of a simple system involving only a few resonances suggest that the unphysical solutions show characteristic correlations (pseudoresonances) in different partial waves near a true resonance in one partial wave. Therefore, the physical solution requires the fewest resonant states. We apply this criterion to an analysis of $^{16}O(d,\alpha_1)^{14}N$.

Even for purely elastic scattering a phase-shift analysis using only differential cross sections at one energy will have several solutions all of which give identically the same cross sections at all angles. An example is the Minami ambiguity[1] for elastic scattering of spin-$\frac{1}{2}$ by spin-0 particles where the interchange of all pairs of phase shifts with the same J but different $l=J\pm\frac{1}{2}$ leaves the angular distribution unaltered. Several authors have discussed the multiplicity of such ambiguities for cases of elastic scattering[2-4] and reactions.[5] Gersten[4] has, in addition, given a prescription for finding the entire set of solutions given one of them.

The selection of the correct solution from the ambiguous set is possible in certain cases. For elastic scattering of charged particles the long-range Coulomb interaction defines the phase shifts for high l. Because these would be altered in the transformation to other solutions, the ambiguities cannot exist. Polarization measurements for purely elastic spin-$\frac{1}{2}$–spin-0 scatter-