clusions. First, curves of KCl F center versus dose obtained by making optical absorption measurements during irradiation at 85°K do not differ qualitatively from curves made by traditional techniques. To a large extent, this agreement can be attributed to the fact that the F-center coloring is almost completely stable at this temperature. A study of the room-temperature data[5,6] shows that these results could not have been reliably established unless, as in this instance, absorption measurements can be made both during and after irradiation. Second, they can be unequivocally resolved into one linear and one saturating-exponential component. In retrospect, this separation into components could have been surmised from previous publications[1-3] although the data they contain are too sparse to establish this analysis reliably. Third, the accurate curves of F center versus time obtained at precisely controlled dose rates leads to an entirely new result: The constants a_1 and α_L in the expression $\alpha(t)=A_1[1-\exp(-a_1 t)]+\alpha_L t$ are proportional to the dose rate, and A_1 is independent of the dose rate. Fourth, these observations are in accord with a phenomenological derivation, describing the growth of color centers during irradiation, which appears to be the simplest conceivable theory describing all the results. In this model both the precursor formation rate and the rate that precursors are converted into color centers increases linearly with dose rate.

Finally, it must be emphasized that these results were obtained with samples cleaved from just one large crystal. If different crystals are used, which will presumably have a different set of impurities, or the experimental conditions are changed, e.g., by straining the crystals before irradiation or changing the temperature, different growth curves may be encountered. Measurements on KCl crystals from other sources and using different experimental conditions will be reported in the future.

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*Research supported by the U. S. Atomic Energy Commission.

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Exact Solution for a Closed Chain of Classical Spins with Arbitrary, Isotropic Nearest-Neighbor Exchange

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(Received 19 April 1971)

The problem of a system of classical spins of arbitrary dimensionality interacting through an arbitrary, isotropic nearest-neighbor–exchange Hamiltonian is solved exactly for the closed linear chain. The $2m$-uple spin-correlation function for spins a distance k apart is shown to be the kth power of the ratio of the mth eigenvalue to the largest one. It is proposed that experimental data be used to find the general functional form of the exchange interaction.

Stanley[1] has shown that a system of classical spins of arbitrary dimensionality located on a linear chain (Bethe lattice) is exactly solvable for the isotropic bilinear exchange interaction. In a previous paper,[2] we generalized Stanley's result to include an arbitrary biquadratic interaction term. With the aid of the Funk-Hecke theorem,[3] it is now shown that this system is actually solvable for any isotropic interaction of the form $H_i^{(v)}=-f(\vec{S}_i\cdot\vec{S}_{i+1})$, where $f(x)$ is any function of the real variable x, bounded and continuous for $|x|\leqslant 1$. Stanley *et al.*[4] made use of this theorem to show that the bilinearly interacting ring provides additional evidence in support of the conjecture of eigenvalue degeneracy as a possible mathematical mechanism for a phase transition. Our results show that this mechanism is not restricted to the bilinear interactions, nor is it limited to dipolar-type phase transitions.

Consider a closed chain of N classical spins

$\vec{S}_i$ of arbitrary dimensionality ν which have isotropic nearest-neighbor interactions described by the Hamiltonian

$$H^{(\nu)}=\sum_{i=1}^{N}H_i^{(\nu)}=-\sum_{i=1}^{N}f(\vec{S}_i\cdot\vec{S}_{i+1}),\tag{1}$$

with the constraints that $\vec{S}_i^2=1$, $\vec{S}_{N+1}=\vec{S}_1$, and with $f(x)$ any function bounded and continuous in $|x|\leqslant 1$. The partition function of this system is then

$$Q_N=\omega^{-N}\int\cdots\int d\Omega_1\cdots d\Omega_N\prod_{i=1}^{N}\exp[\beta f(\vec{S}_i\cdot\vec{S}_{i+1})]=\int d\Omega_1 K^{(N)}(\vec{S}_1,\vec{S}_1),\tag{2}$$

where $d\Omega_i$ denotes an element of solid angle in the ν-dimensional space of $\vec{S}_i$, $\beta=(k_BT)^{-1}$, $\omega=2\pi^{\nu/2}/\Gamma(\frac{1}{2}\nu)$ is the area of the unit hypersphere, and $K^{(p)}(\vec{S}_i,\vec{S}_j)$ denotes the kernel of pth iterate of the integral operator with kernel $K(\vec{S}_i,\vec{S}_j)=\omega^{-1}\exp[\beta f(\vec{S}_i\cdot\vec{S}_j)]$. According to the Funk-Hecke theorem, a complete set of eigenfunctions $\psi_{nl}(\vec{S})$ for this kernel are the hypersurface harmonics of order $\frac{1}{2}(\nu-2)$, with corresponding discrete eigenvalues

$$\lambda_{nl}=\lambda_n=\frac{(4\pi)^{(\nu-2)/2}n!\,\Gamma(\frac{1}{2}(\nu-2))}{\Gamma(n+\nu-2)}\omega^{-1}\int_{-1}^{1}dx\,e^{\beta f(x)}C_n^{(\nu-2)/2}(x)(1-x^2)^{(\nu-3)/2},\tag{3}$$

where $C_n^{(\nu-2)/2}$ is the Gegenbauer polynomial.[5] It is well known[6] that $K^{(p)}$ can be expressed in the form

$$K^{(p)}(\vec{S}_i,\vec{S}_j)=\sum_{n=0}^{\infty}\sum_{l=1}^{h(n,\nu-2)}\lambda_n^{\,p}\psi_{nl}^*(\vec{S}_i)\psi_{nl}(\vec{S}_j),\tag{4}$$

where $h(n,\nu-2)=(2n+\nu-2)(n+\nu-3)!/n!(\nu-2)!$ is the degeneracy of λ_n. It then follows that the partition function can be written as

$$Q_N=\sum_{n=0}^{\infty}\lambda_n^{\,N}h(n,\nu-2),\tag{5}$$

and further, that the thermal average of any function $A(\vec{S}_i,\vec{S}_{i+k})$ of spins separated by a distance k can be expressed in the form

$$\begin{aligned}\langle A_k\rangle &\equiv Q_N^{-1}\mathrm{Tr}[A(\vec{S}_i,\vec{S}_{i+k})e^{-\beta H}]\\ &=Q_N^{-1}\sum_{nn'll'}\lambda_n^{\,N-k}\lambda_{n'}^{\,k}I(n,l;n',l'),\end{aligned}\tag{6}$$

where

$$I(n,l;n',l')=\iint d\Omega_i d\Omega_{i+k}\,A(\vec{S}_i,\vec{S}_{i+k})\psi_{nl}^*(\vec{S}_i)\psi_{n'l'}(\vec{S}_i)\psi_{nl}(\vec{S}_{i+k})\psi_{n'l'}^*(\vec{S}_{i+k}).\tag{7}$$

It is now easily shown that, as long as $\exp[\beta f(x)]$ is bounded for $|x|\leqslant 1$,

$$\lambda_0>\lambda_n \text{ for all } n\geqslant 1,\tag{8}$$

so that, as $N\to\infty$,

$$Q_N=\lambda_0^{\,N};\quad \langle A_k\rangle=\sum_{n'=0}^{\infty}\sum_{l'=1}^{h(n',\nu-2)}\left(\frac{\lambda_{n'}}{\lambda_0}\right)^k I(0,l;n',l').\tag{9}$$

If one now defines a $2m$-uple spin-correlation function $Y_m^{(\nu)}(k)$ as the thermal average of $A(\vec{S}_i,\vec{S}_{i+k})=C_m^{(\nu-2)/2}(\vec{S}_i\cdot\vec{S}_{i+k})/C_m^{(\nu-2)/2}(1)$ and makes use of the orthonomality property of the Gegenbauer polynomials, Eq. (7) yields the results that $I(0,l;n',l')=\delta_{n'm}/h(m,\nu-2)$. Hence the $2m$-uple spin-correlation function of two spins separated by a distance k can, in general, be written in the form

$$Y_m^{(\nu)}(k)=(Y_m^{(\nu)})^k=(\lambda_m/\lambda_0)^k,\tag{10}$$

regardless of spin dimensionality and the explicit form of the nearest-neighbor interaction function $f(x)$. If we were now to set $f(x)=J_1x+J_2x^2$, then Q_N, $Y_1^{(\nu)}$, and $Y_2^{(\nu)}$ would reduce to our previous results.[2]

It follows from Eq. (8) that since λ_m/λ_0 can never equal 1, long-range order can never occur at any finite temperature. When $T\to 0$, if the absolute maximum of $f(x)$ occurs at $|x_0|=1$, every kind of long-range order occurs simultaneously; otherwise $(-1<x_0<1)$ it is impossible to develop any kind of long-range order, even at $T=0$.[2]

It is now well known that there exist a number of real physical systems for which the dominant interaction is along a linear chain.[7] Furthermore, for many substances with $S>\frac{1}{2}$, the classical limit $(S\to\infty)$ appears to yield a valid description above the ordering temperature (which is zero for the present situation!).[8] Since the quantity $Y_m^{(\nu)}$ is simply related to a measurable quantity,[2,7] the present results open up the possibility of using data of $Y_m^{(\nu)}$ vs β to invert Eqs. (3) and (10) to attempt an explicit determination of the general functional form $f(x)$. For $m=1$ (the usual dipolar case), given $Y_1^{(\nu)}(\beta)$, one would then have to "simply" invert the equation

$$Y_1^{(\nu)}(\beta)=\frac{\int_{-1}^{1}dx\,x(1-x^2)^{(\nu-3)/2}e^{\beta f(x)}}{\int_{-1}^{1}dx(1-x^2)^{(\nu-3)/2}e^{\beta f(x)}} \tag{11}$$

to get $f(x)$.

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$$\lim_{\alpha\to0}\alpha^{-1}C_n^{(\alpha)}(x)=2n^{-1}T_n(x).$$

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Suggestion for Choosing the Single-Particle Energies in Doubly Closed-Shell Nuclei*

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(Received 10 March 1971)

The single-particle energies in a nucleus of $A\pm1$ particles as taken from experiment are seen to be shifted when used for the calculation of the A-particle–system excited states. This shift can be described as a change in the particle-hole gap arising from an isospin-isospin interaction of the excited nucleon with the remaining core of $A-1$ particles. It lowers the $T=0$ states of ^{40}Ca by about 1.5 MeV and of ^{16}O by 3.75 MeV, and it raises the $T=1$ states one third this amount. This shift gives an account of discrepancies which are observed in all existing Tamm-Dancoff or random-phase approximation calculations of doubly closed shell nuclei.

A large number of calculations[1-8] have already been performed on doubly closed-shell nuclei. The spirit of all these works is essentially the same. One chooses a single-particle basis and splits the Hamiltonian H of the system into two parts: H_0, which is already diagonal for the particle-hole excitations $|j'j^{-1}\rangle$, and a residual interaction V. The eigenvalues of H_0 are generally taken from the experimental data on the neighboring $A\pm1$ nuclei. The matrix elements $\langle(j'j^{-1})JT|\times V|(j'j^{-1})JT\rangle$ have a component diagonal in the projection quantum numbers of the particle and the hole (and thus independent of J). This component is a correction to the particle-hole energy $\epsilon_{j'}-\epsilon_j$:

$$\langle j'm'jm|V|j'm'jm\rangle, \tag{1}$$

schematized by the graph of Fig. 1(a). In principle, it takes into account the change in the single-particle energy of a nucleon which interacts

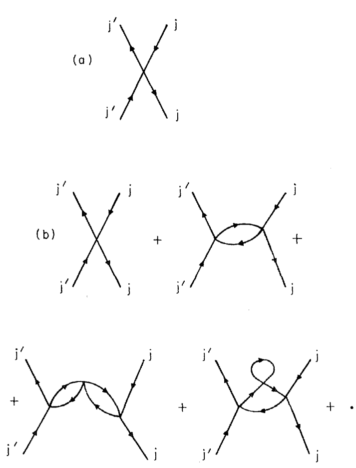

FIG. 1. (a) The correction to the single-particle energies (taken from the $A\pm1$ nuclei) usually included in a TDA or RPA calculation. (b) This same correction as a sum, calculated to all orders in the nucleon-nucleon interaction.