$\times 10^{-3}$ which is roughly the threshold for the curves in Fig. 2. Note that the average error field is ~0.10 of the peak error field. As the error field increases, the measured plasma loss increases as $(B)^{2.2}$ for the $I_z = 4000$ A case which is predicted by Eq. (1). At large error fields the observed loss in the sheared fields approaches the loss for the unsheared case. To explain this behavior we assume that electric fields again develop parallel to the field lines at large field errors and cause increased ion loss. The plasma loss was observed to increase as the background pressure was decreased at zero applied error and zero B_θ. This is in agreement with this model for loss due to field errors and is different from the case[11] for support induced loss in a sheared field where decreasing pressure decreased the hoop loss.

The results of these experiments and their interpretation indicate that the observed anomalous loss of plasma to the internal hoops of existing multipoles can be caused by small (<1%) field errors. Since present construction techniques cannot produce a magnetic field of arbitrarily small field errors, a toroidal field can be used to reduce the effects of the errors to a tolerable level.

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Thermodynamics of the Heisenberg-Ising Ring for $\Delta \geqslant 1$*

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The thermodynamics of the Heisenberg-Ising ring is reduced to the solution of a system of recurrent nonlinear integral equations.

The energy levels E of the Hamiltonian[1]

$$H = \sum_{n=1}^{N} S_{n,x}S_{n+1,x} + S_{n,y}S_{n+1,y} + \Delta(S_{n,z}S_{n+1,z} - \tfrac{1}{4}), \quad (1)$$

with $S_N \equiv S_1$, are given by the coupled equations (using Bethe's notation)[2]

$$E = \sum_{\alpha=1}^{N} (\cos k_\alpha - \Delta), \quad (2)$$

$$N k_\alpha = 2\pi\lambda_\alpha + \sum_{\beta=1}^{M}{}' \psi_{\alpha\beta}, \quad \alpha = [1, M], \quad (3)$$

where the pseudomomenta k_α and antisymmetric phases $\psi_{\alpha\beta}$ are parametrized[3] in terms of auxiliary quantities φ_α:

$$\cot\tfrac{1}{2}k_\alpha = \coth\tfrac{1}{2}\Phi \tan\tfrac{1}{2}\varphi_\alpha, \quad \Delta = \cosh\Phi \geqslant 1, \quad (4)$$

$$\cot\tfrac{1}{2}\psi_{\alpha\beta} = \coth\Phi \tan\tfrac{1}{2}(\varphi_\alpha - \varphi_\beta), \quad (5)$$

$$0 \leqslant k_\alpha \leqslant 2\pi, \quad -\pi \leqslant \psi_{\alpha\beta} \leqslant \pi, \quad -\pi \leqslant \varphi_\alpha \leqslant \pi. \quad (6)$$

The integer M is related to the magnetic spin component $S_z = \frac{1}{2}N - M$. So far the appropriate sets of integers λ are only known for the restricted class of states for which the momenta are real numbers.[4]

Applying Bethe's method,[2] it can be shown in the limit of large N that the roots of Eq. (3) in the complex plane of parameters φ_μ are grouped in various strings characterized by a common real abscissa φ and an order n. Let us call such a string a *complex*[2] $C_n(\varphi)$. We have that $C_n(\varphi)$ is comprised of the set

$$\varphi_\mu = \varphi + i\mu\Phi + i\delta_\mu\Phi, \tag{7}$$

$$\mu = -(n-1), -(n-3), \cdots, n-3, n-1, \tag{8}$$

where δ_μ is an exponentially small quantity:

$$\ln(1/\delta_\mu) = O(N). \tag{9}$$

Suppose we have ν_n complexes of order n with real abscissas $\varphi_{n,i}$, $i = [1, \nu_n]$. The asymptotic form of the system (3) can be written as a system of equations for the real $\varphi_{n,i}$:

$$Nf_n(\varphi_{n,i}) = 2\pi J_{n,i} + \sum_{m,p} \sum_{j=1}^{\nu_m} [nmp] f_p(\varphi_{n,i} - \varphi_{m,j})$$
$$n = 1, 2, 3, \cdots, \quad i = 1, 2, \cdots, \nu_n, \tag{10}$$

with $f_n(\varphi)$ a continuous odd function defined by

$$\tan\tfrac{1}{2}f_n(\varphi) = -\coth\tfrac{1}{2}n\Phi \tan\tfrac{1}{2}\varphi, \tag{11}$$

and $[nmp]$ the completely symmetric symbol given by

$$[nmp] = \begin{cases} 1 & p = |m \pm n|, \\ 2 & |m-n| < p < m+n, \quad m+n+p \text{ even}, \\ 0 & \text{otherwise}. \end{cases} \tag{12}$$

By using a continuity principle in Δ and making a noncrossing hypothesis for the real $\varphi_{n,i}$, the study of the limit $\Delta \to \infty$ on the wave function of Eq. (10) permits us to reach the following conclusion: For each order n, the quantum numbers $J_{n,i}$ are integers (or half-integers) forming an increasing sequence on some lattice interval depending on n. This fact allows a fermion-like description of the set of bound states or complex of each order.

Following the method devised by Yang and Yang[5] for the thermodynamics of the one-dimensional boson system, we express the energy and entropy of the infinite ring as a functional of the density of complex $C_n(\varphi)$: "particle" density $\rho_n(\varphi)$ and "hole" density $\tilde{\rho}_n(\varphi)$. Equations (10) simply give the relations

$$\rho_n + \tilde{\rho}_n = \mathrm{dn} * (\tilde{\rho}_{n+1} + \tilde{\rho}_{n-1}), \quad n \geq 1,$$

with the convention $\tilde{\rho}_0(\varphi) = 2\pi\delta(\varphi)$ and the notation

$$(\mathrm{dn} * \rho)(\varphi) = (2\pi)^{-1} \int_{-\pi}^{+\pi} \mathrm{dn}(\varphi - \varphi')\rho(\varphi') d\varphi'. \tag{13}$$

The elliptic Jacobi function[6] $\mathrm{dn}(\varphi)$ has the periods 2π and $4i\Phi$. In the limit $\Delta \to 1$ $(\Phi \to 0)$, it is sufficient to replace $\mathrm{dn}(\varphi)d\varphi$ by

$$(\cosh\tfrac{1}{2}\pi x)^{-1} \tfrac{1}{2}\pi dx, \quad x = \varphi/\Phi. \tag{14a}$$

We find that the energy per site is

$$E/N = E_0/N + \sinh\Phi \int_{-\pi}^{+\pi} \mathrm{dn}(\varphi)\tilde{\rho}_1(\varphi) d\varphi, \tag{15}$$

where E_0 is the energy of the antiferromagnetic ground state,[7] and the entropy per site is

$$S/N = \sum_{n>0} \int_{-\pi}^{+\pi} d\varphi [(\rho_n + \tilde{\rho}_n) \ln(\rho + \tilde{\rho}_n) - \rho_n \ln\rho_n - \tilde{\rho}_n \ln\tilde{\rho}_n]. \tag{16}$$

Minimizing the free energy

$$F = E - TS, \tag{17}$$

with the constraint,

$$\sigma = S_z/N = \tfrac{1}{2} - M/N, \tag{18}$$

we obtain the equilibrium density of the complex in the form

$$\rho_n(\varphi) = \frac{r_n(\varphi)}{1 + \exp[\epsilon_n(\varphi)/T]}, \quad \tilde{\rho}_n(\varphi) = \frac{r_n(\varphi)}{1 + \exp[-\epsilon_n(\varphi)/T]} \tag{19}$$

where the *pseudoenergies* $\epsilon_n(\varphi)$ are given by a recurrent set of nonlinear integral equations:

$$\epsilon_n/T = \mathrm{dn}*\{\ln([1+\exp(\epsilon_{n+1}/T)][1+\exp(\epsilon_{n-1}/T)])\}, \quad n>1,$$
$$\epsilon_1/T = \mathrm{dn}*\{\ln[1+\exp(\epsilon_2/T)]\} - (T^{-1}\sinh\Phi)\mathrm{dn}. \tag{20}$$

These equations must be completed by the following asymptotic condition:

$$\lim_{n\to\infty} T^{-1}\epsilon_n(\varphi)/n = \lambda > 0. \tag{21}$$

This likely determines the whole set of ϵ_n as functions of φ, T, and λ. The parameter λ will be determined by the magnetization constraint.

Now the quantities $r_n = \rho_n + \tilde{\rho}_n$ are given by

$$r_n = +\frac{1}{2\pi}\frac{T^2}{\sinh\Phi}\frac{\partial}{\partial T}\left(\frac{\epsilon_n}{T}\right)\bigg|_\lambda; \tag{22}$$

the free energy per site is

$$\frac{F}{N} = \frac{E_0}{N} - \frac{T}{2}\int_{-\pi}^{+\pi} d\varphi\, \mathrm{dn}(\varphi)\ln[1+\exp(\epsilon_1/T)] + \sigma\lambda T; \tag{23}$$

and the magnetization per site is

$$\sigma = (1/2\pi T)\int_{-\pi}^{+\pi} d\varphi\, \mathrm{dn}(\varphi)(\partial\epsilon_1/\partial\lambda)[1+\exp(-\epsilon_1/T)]^{-1}. \tag{24}$$

The relation $\partial F/\partial\lambda|_\sigma = 0$ or

$$\lambda T = \partial(F/N)/\partial\sigma|_T = H_0 \tag{25}$$

permits us to interpret λT as the magnetic field H_0 in presence of which the magnetization has the value σ.

The study of the Ising limit ($\Delta\to\infty$, Δ/T finite) provides a check of the calculation. The recurrence relation (20) can be solved in this case and leads to the correct thermodynamic functions of the Ising model in one dimension. In the zero-temperature limit, $T=+0$, the results of Griffiths[4] and Yang[1] are obtained. The limiting $|\epsilon_n|$ can be interpreted as an elementary excitation energy above the Fermi level H_0. In particular the minimum magnetic field H_1 giving a nonzero magnetization is

$$\lim_{\sigma\to 0} H_0 \equiv H_1 = 2\sinh\Phi\,\mathrm{dn}(\pi), \tag{26}$$

which is precisely the value of the energy gap[8,9] between the antiferromagnetic ground state and the first excited state $S_z=1$.

In the limit $T=+0$ and $\sigma=+0$, we obtain

$$\epsilon_n = H_1 n - H_1, \quad n>1; \quad -\epsilon_1 = (\text{energy of spin wave } S_z=1) - H_1. \tag{27}$$

Finally the limit $\Delta = 1+0$ can be easily written by making the substitution (14a) in all the equations; for instance Eq. (20) becomes

$$\frac{\epsilon_n(x)}{T} = \frac{1}{4}\int_{-\infty}^{+\infty}\frac{dx'}{\cosh|\frac{1}{2}\pi(x-x')|}\ln\left\{\left[1+\exp\left(\frac{\epsilon_{n+1}(x')}{T}\right)\right]\left[1+\exp\left(\frac{\epsilon_{n-1}(x')}{T}\right)\right]\right\}, \tag{28}$$

which is equivalent to the set of equations recently found by Takahashi[10] in his solution of the same problem for $\Delta=1$.

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Toroidal Contributions to the Shear and the $j_\parallel$-Kink Instability in the Tokamak and Screw Pinch

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It is shown that, for a Tokamak plasma with fixed boundary, the toroidal contributions to the shear are sufficiently important to modify the $j_\parallel$-kink instability. Assuming that β is small and using the energy principle, δW is calculated to order $(r/R)^6$ and the toroidal corrections are shown to change the sign of the destabilizing term. One consequence is that the range of rotational transform angles for which the $m=1$, $\nu=1$ instability will occur, when the shear is weak, is below the Kruskal-Shafranov limit and not above it. Also, the new terms have an important stabilizing effect for the screw pinch.

In the studies of the magnetohydrodynamic (MHD) stability of Tokamak plasmas with conducting-wall boundaries,[1-3] use has been made of the energy principle and also of a series expansion taking the aspect ratio r/R as a small parameter. δW has been calculated to order $(r/R)^4$. To this order, the only possible destabilizing terms are proportional to the pressure gradient; these appear if $|dp/dr| \gtrsim (r/R)^2(B^2/r)$, that is, if $\beta \sim (r/R)^2$. Because $B_\varphi/B_\theta \sim R/r$, the destabilizing term due to $j_\parallel$, which is present in linear-pinch theory, is of higher order. The only toroidal contributions related to $j_\parallel$ $O((r/R)^4)$ arise from the part of $j_\parallel$ associated with the nonzero divergence of $j_\perp$ and are proportional to dp/dr. (The same symbols and coordinate system are used as in Ref. 1. In particular, θ and φ are the angular coordinates in the poloidal and toroidal directions, respectively. The subscripts, $\parallel$ and $\perp$, denote vector components parallel to $\vec{B}$ and $\nabla p \times \vec{B}$, respectively.)

Previous studies of the $j_\parallel$-kink instability have considered only the linear pinch and the linear stellarator. For the linear pinch with $dp/dr = 0$ and assuming $B_\theta/B_z \sim r/R$ (giving field components similar in magnitude to those of a Tokamak), δW, for orders up to $(r/R)^6$ and after minimizing with respect to the θ and z components of the plasma displacement $\xi \exp[i(m\theta - k_z z)]$, is

$$\delta W = \int \tfrac{1}{2}\pi r dr[\delta B_r^2 + \delta B_\theta^2 + (rk_\parallel B\xi_r^2/m)dj_\parallel/dr], \tag{1}$$

where $k_\parallel B = (m/r)B_\theta - k_z B_z$. The third term is the only term that can be negative and wrongly suggests that large radial gradients in $j_\parallel$ must be destabilizing. (This mistake was made by Kadomtsev and Pogutse.[4]) In fact, on substituting for δB_r and δB_θ and integrating the δB_θ^2 term by parts, assuming perfectly conducting plasma exists at all radii out to the conducting wall, most of the $dj_\parallel/dr$ term is canceled and δW reduces to[5]

$$\delta W = \int \tfrac{1}{2}\pi r dr\{(k_\parallel^2 B^2/m^2)[(m^2-1)\xi_r^2 + r^2(\partial\xi_r/\partial r)^2] - 4k_\parallel B B_\theta k_z^2 r \xi_r^2 m^{-3}\}. \tag{2}$$

k_z is $O(r/R)$ and hence, for instability, it follows that $k_\parallel$ must be positive and $O((r/R)^3)$. (This choice of $k_\parallel$ is possible if the shear is assumed weak.) The net destabilizing term, the last term in the square brackets, is then $O((r/R)^6)$. It is thus small and is not proportional to $dj_\parallel/dr$. For unfavorable shear conditions, such as a maximum in the rotational transform as described in Refs. 5 and 11, there will be instability with growth rate $O((r/R)^3(B^2/\rho)^{1/2})$, where ρ is the plasma density.