lations are a useful spectroscopic tool even in cases where the decay is to a state of nonzero spin. Plans to extend measurements of this kind to other highly excited states produced in heavy-ion-induced reactions are under way.

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New Model for the Interaction Between a Moving Charged Particle and a Dielectric, and the Cherenkov Effect*

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A new model is described for the interaction between a moving charged particle executing any two-dimensional motion and an infinite, homogeneous, linear dielectric. The model is based on the exact solution of the self-consistent integro-differential equation of molecular optics. The main results are illustrated by application to the Cherenkov effect.

Consider first a particle of charge e that moves on a trajectory $\vec{r}=\vec{r}_e(t)$ in the plane $z=0$ *in vacuo*. The Fourier frequency transforms [with kernel $(1/2\pi)\exp(i\omega t)$] of the electric and the magnetic fields that are generated by the particle may readily be derived by the use of the retarded potentials. It was shown by Asby and Wolf[1] that in each half-space $z\gtrless 0$ the fields have the following angular spectrum representation:

$$\vec{E}^{(v)}(\vec{r},\omega)=\iint_{-\infty}^{+\infty}\vec{e}^{(v)}(p,q;\omega;\gtrless)\exp[ik(px+qy\pm mz)]dpdq, \quad (1)$$

$$\vec{H}^{(v)}(\vec{r},\omega)=\iint_{-\infty}^{+\infty}\vec{h}^{(v)}(p,q;\omega;\gtrless)\exp[ik(px+qy\pm mz)]dpdq. \quad (2)$$

Here

$$k=\omega/c, \quad (3)$$

$$m=\begin{cases}+(1-p^2-q^2)^{1/2} \text{ if } p^2+q^2\leq 1, & (4a)\\ +i(p^2+q^2-1)^{1/2} \text{ if } p^2+q^2\geq 1; & (4b)\end{cases}$$

and the complex spectral vector amplitudes are given by

$$\vec{e}^{(v)}(p,q;\omega;\gtrless)=\hat{s}^{(\pm)}\times[\hat{s}^{(\pm)}\times\vec{f}(p,q;\omega)], \quad (5)$$

$$\vec{h}^{(v)}(p,q;\omega;\gtrless)=-\hat{s}^{(\pm)}\times\vec{f}(p,q;\omega), \quad (6)$$

with

$$\vec{f}(p,q;\omega)=(e/cm)(k/2\pi)^2\int_{-\infty}^{+\infty}\vec{V}(t')\exp\{i[\omega t'-k(px_e'+qy_e')]\}dt'. \quad (7)$$

In Eq. (7) x_e', y_e' are the coordinates and $\vec{V}(t')$ the velocity of the charged particle at time t'; c is the speed of light *in vacuo*. In (5) and (6), $\hat{s}^{(\pm)}$ are the unit vectors $(p,q,\pm m)$; the positive or negative sign on $\hat{s}^{(\pm)}$ and in $\pm m$ in (1) and (2), and the symbol > or < in the arguments of $\vec{e}^{(v)}$ and $\vec{h}^{(v)}$, are taken according as the field point $\vec{r}$ is in the half-space $z>0$ or $z<0$, respectively.

Equations (1) and (2) are exact mode expansions of the fields. Each equation expresses the Fourier frequency transform of the fields as a superposition of plane waves, all of the *same* wave number $k = \omega/c$ appropriate to the particular frequency component ω of the fields. Each mode is labeled by the pair of parameters p, q and the frequency ω, and consists of a plane wave

$$\vec{e}^{(v)}(p, q; \omega; >) \exp[ik(px + qy + mz)] \text{ if } z > 0,$$

or

$$\vec{e}^{(v)}(p, q; \omega; <) \exp[ik(px + qy - mz)] \text{ if } z < 0.$$

In view of the definition of m [Eqs. (4)] we see that if $p^2 + q^2 \leq 1$ the waves are ordinary homogeneous waves propagated away from the plane of motion of the particle, and that if $p^2 + q^2 > 1$ they are evanescent waves propagated in directions parallel to the plane of motion of the particle ($z = 0$) and decaying exponentially in amplitude with increasing distance $|z|$ from that plane. It may readily be shown by examining the asymptotic behavior of (1) and (2) as $kr \to \infty$ in any fixed direction that only the homogeneous waves in the angular spectrum give rise to radiation.[2]

Suppose now that the charged particle executes the same motion in an infinite dielectric, which we assume to be linear, homogeneous, isotropic, and nonmagnetic. Then according to molecular optics,[3,4] the average effective fields $\vec{E}'(\vec{r}, \omega)$ and $\vec{H}'(\vec{r}, \omega)$ at the point $\vec{r}$ in either of the two half-spaces $z > 0$ or $z < 0$ may be shown to satisfy the equations

$$\vec{E}'(\vec{r}, \omega) = \vec{E}^{(v)}(\vec{r}, \omega) + N\alpha(\omega)\int \nabla\times\nabla\times[\vec{E}'(\vec{r}', \omega) G_\omega(|\vec{r}-\vec{r}'|)]d^3r', \tag{8}$$

$$\vec{H}'(\vec{r}, \omega) = \vec{H}^{(v)}(\vec{r}, \omega) - ikN\alpha(\omega)\int \nabla\times[\vec{E}'(\vec{r}', \omega) G_\omega(|\vec{r}-\vec{r}'|)]d^3r', \tag{9}$$

where the integration extends over the two half-spaces $z > 0$ and $z < 0$, except for a vanishingly small sphere centered on the field point at $\vec{r}$. In these equations $\vec{E}^{(v)}$ and $\vec{H}^{(v)}$ are, of course, the vacuum fields, given by Eqs. (1) and (2); N is the average number of molecules per unit volume; $\alpha(\omega)$ is the average polarizability (at frequency ω) of a molecule; and

$$G_\omega(|\vec{r}-\vec{r}'|) = |\vec{r}-\vec{r}'|^{-1}\exp(ik|\vec{r}-\vec{r}'|) \tag{10}$$

is the outgoing free-space Green's function.

Now according to a mathematical lemma [Ref. 3, Eq. (11) of Appendix V], the integral in (8) may be rewritten as

$$\int \nabla\times\nabla\times[\vec{E}'(\vec{r}', \omega)G_\omega(|\vec{r}-\vec{r}'|)]d^3r' = \nabla\times\nabla\times\int \vec{E}'(\vec{r}', \omega)G_\omega(|\vec{r}-\vec{r}'|)d^3r' - (8\pi/3)\vec{E}'(\vec{r}, \omega). \tag{11}$$

On the other hand one can show that in the integral in (9) the operator $\nabla\times$ may be taken outside the integral sign. Further, one has the Lorentz relation between the effective electric field $\vec{E}'$ and the macroscopic Maxwell electric field $\vec{E}^{(d)}$ in the dielectric:

$$\vec{E}'(\vec{r}, \omega) = \vec{E}^{(d)}(\vec{r}, \omega) + (4\pi/3)N\alpha(\omega)\vec{E}'(\vec{r}, \omega). \tag{12}$$

Since the dielectric was assumed to be nonmagnetic, the effective magnetic field is equal to the macroscopic Maxwell field $\vec{H}^{(d)}$ in the dielectric:

$$\vec{H}'(\vec{r}, \omega) = \vec{H}^{(d)}(\vec{r}, \omega). \tag{13}$$

Finally, we also have the Lorentz-Lorenz relation connecting the molecular polarizability per unit volume, $N\alpha$, with the refractive index n of the medium:

$$[n^2(\omega) - 1]/[n^2(\omega) + 2] = (4\pi/3)N\alpha(\omega). \tag{14}$$

Using Eqs. (11)-(14), Eqs. (8) and (9) may be rewritten as the following relations between the Maxwell vacuum fields and the fields in the dielectric:

$$\vec{E}^{(d)}(\vec{r}, \omega) = \frac{1}{n_2}\vec{E}^{(v)}(\vec{r}, \omega) + \frac{1}{4\pi}\left[\frac{n^2(\omega)-1}{n^2(\omega)}\right]\nabla\times\nabla\times\int \vec{E}^{(d)}(\vec{r}', \omega)G_\omega(|\vec{r}-\vec{r}'|)d^3r', \tag{15}$$

$$\vec{H}^{(d)}(\vec{r}, \omega) = \vec{H}^{(v)}(\vec{r}, \omega) - (ik/4\pi)[n^2(\omega)-1]\nabla\times\int \vec{E}^{(d)}(\vec{r}', \omega)G_\omega(|\vec{r}-\vec{r}'|)d^3r'. \tag{16}$$

Equation (15) is an integro-differential equation for the Maxwell electric field in the dielectric in terms of the Maxwell vacuum electric field. We have succeeded in obtaining the exact solution of Eq. (15), and we will describe in another publication the technique we used. Here we only wish to report the physical implications of the solution. For this purpose we represent $\vec{E}^{(d)}(\vec{r}, \omega)$ and $\vec{H}^{(d)}(\vec{r}, \omega)$ as angular spectra of plane waves, all with the wave number $n(\omega)k = n(\omega)\omega/c$ appropriate to waves of frequency ω in the dielectric. In particular, the representation of $\vec{E}^{(d)}$ has the form

$$\vec{E}^{(d)}(\vec{r}, \omega) = \iint_{-\infty}^{+\infty} \vec{e}^{(d)}(p', q'; \omega; \gtrless) \exp[in(\omega)k(p'x + q'y \pm m'z)]dp'dq'. \quad (17)$$

Here m' bears the same relation to p' and q' as m bears to p and q [cf. Eqs. (4)]. The solution of the integro-differential equation (15) then implies the following: *The field* $\vec{E}^{(d)}(\vec{r}, \omega)$ *in the dielectric is obtained from the vacuum field* $\vec{E}^{(v)}(\vec{r}, \omega)$ *by a transformation in which each mode* (p, q, ω) *of the vacuum field is transformed into one and only one mode* (p', q', ω) *of the field in the dielectric, with*

$$p \to p' = p/n(\omega), \quad q \to q' = q/n(\omega). \quad (18)$$

Moreover, the transformation law for the vector amplitudes $[\vec{e}^{(v)} \to \vec{e}^{(d)}]$ is of a very simple form [given by Eqs. (21) below].

We note that (18) implies that for corresponding modes,

$$\begin{vmatrix} 0 & 0 & 1 \\ p & q & m \\ p' & q' & m' \end{vmatrix} = 0, \quad (19)$$

which shows that the unit (real or complex) vectors (p, q, m) and (p', q', m') of corresponding modes are coplanar, with the normal to the plane of motion of the particle (the plane $z = 0$). If the corresponding modes are homogeneous, i.e., if $p^2 + q^2 \leq 1$ and $p'^2 + q'^2 \leq 1$, then $m = \cos\theta_v$, $m' = \cos\theta_d$, where θ_v and θ_d are the angles which the respective directions of propagation of the waves *in vacuo* and in the dielectric make with the normal to the plane of the particle motion (see Fig. 1). Equation (18) together with Eq. (4a) and a similar formula for m' imply that for corresponding modes

$$\sin\theta_v / \sin\theta_d = n(\omega). \quad (20)$$

The relation (20), together with the complanarity condition expressed by (19), is formally identical with the *law of refraction* (Ref. 3, Sect. 1.5.1). Actually the analogy with the problem of refraction goes further. If we resolve the vector amplitudes $\vec{e}^{(v)}$ and $\vec{e}^{(d)}$ into components parallel and perpendicular to the plane containing corresponding wave normals $(p, q, \pm m)$ and $(p', q', \pm m')$, then the solution of Eq. (15) may be shown to imply that for corresponding modes,

$$\frac{e_{\parallel}^{(d)}}{e_{\parallel}^{(v)}} = \frac{\sin\theta_v}{\sin\theta_d}, \quad \frac{e_{\perp}^{(d)}}{e_{\perp}'^{(v)}} = \frac{\sin 2\theta_v}{\sin 2\theta_d}. \quad (21)$$

These relations are of the same mathematical form as, but simpler than, the Fresnel formulas for refraction and reflection (Ref. 3, Sect. 1.5.2).

The results expressed by Eqs. (18) and (21), which are exact within the domain of classical electrodynamics, express basic laws of interaction between the vacuum field of a charged particle executing any two-dimensional motion, and an infinite dielectric.

Several interesting consequences may readily be derived from these laws. We see from (18) that all those modes of the vacuum field for which

$$p^2 + q^2 \leq n^2(\omega) \quad (22)$$

will be transformed upon interaction with the dielectric into homogeneous modes $(p'^2 + q'^2 \leq 1)$. If $n(\omega) > 1$, the domain (22) is seen to include evanescent modes of the vacuum field, namely those modes labeled by p, q, ω, where

$$1 < p^2 + q^2 \leq n^2(\omega). \quad (23)$$

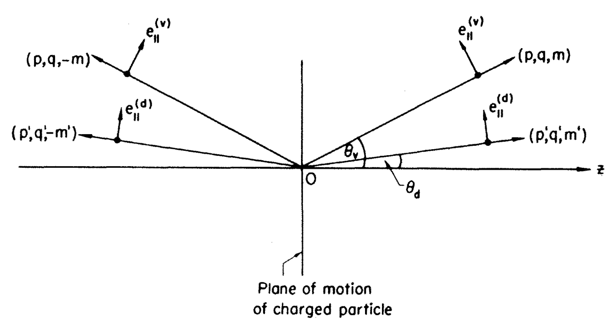

FIG. 1. Illustration of the law of interaction between a moving charged particle and an infinite dielectric. Upon interaction, each (p, q, ω) mode of the vacuum field generated by the moving particle is transformed into a (p', q', ω) mode of the field in the dielectric, where $p' = p/n(\omega)$, $q' = q/n(\omega)$. The components of the electric field amplitudes are transformed according to Eqs. (21). (Only the case when both the modes are homogeneous is illustrated by the figure.)

These evanescent modes will, therefore, upon interaction with the dielectric give rise to radiation. In particular consider a charged particle moving uniformly with speed V in the x direction. From (5) and (7), the spectral vector amplitudes of the vacuum electric field that the charge generates are readily shown to be given by

$$\vec{e}^{(v)}(p, q; \omega; \gtrless) = (ke/2\pi c V m)\hat{s}^{(\pm)} \times [\hat{s}^{(\pm)} \times \vec{V}]\, \delta(p - c/V), \tag{24}$$

where $\delta(x)$ is the Dirac delta function. We see that in this case the parameter p has the sharp value

$$p = p_0 \equiv c/V, \tag{25}$$

but q takes on, of course, all real values. Since the speed V is necessarily smaller than the vacuum speed of light c, p_0 exceeds unity, so that *all* the vacuum modes are now evanescent. The condition (23) for conversion of some of the evanescent modes into homogeneous ones upon interaction with the dielectric now becomes

$$1 \leq (c/V)^2 + q^2 \leq n^2(\omega). \tag{26}$$

The first inequality is automatically satisfied because $c/V > 1$; the second will be satisfied for some q values if and only if $(c/V)^2 < n^2(\omega)$, i.e., if

$$V > c/n(\omega) \equiv v(\omega), \tag{27}$$

where $v(\omega)$ is the phase velocity of the medium for frequency ω. It follows that *if the velocity of the particle exceeds the phase velocity $v(\omega)$ in the medium, those evanescent modes of frequency ω of the vacuum field for which q lies in the range*

$$-n(\omega)\{1 - [v(\omega)/V]^2\}^{1/2} \leq q \leq +n(\omega)\{1 - [v(\omega)/V]^2\}^{1/2} \tag{28}$$

(and for which p has, of course, the sharp value $p_0 = c/V$) *will, upon interaction with the dielectric, give rise to radiation.* The radiation field in the dielectric will then, in view of (18), be created by homogeneous plane waves (p', q', ω), where

$$p' = v(\omega)/V, \quad -\{1 - [v(\omega)/V]^2\}^{1/2} \leq q' \leq \{1 - [v(\omega)/V]^2\}^{1/2}. \tag{29}$$

The first relation of (29) implies that the wave normals of all the homogeneous waves of frequency ω in the dielectric that form the radiation field lie on a cone with semi-angle $\alpha = \cos^{-1}[v(\omega)/V]$ about the direction of propagation of the particle.

Equation (27) is precisely the condition for the generation of Cherenkov radiation,[5] and it is clear that the application of the general results reported in the earlier part of this note would lead to the full description of the Cherenkov field, both above and below threshold.

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