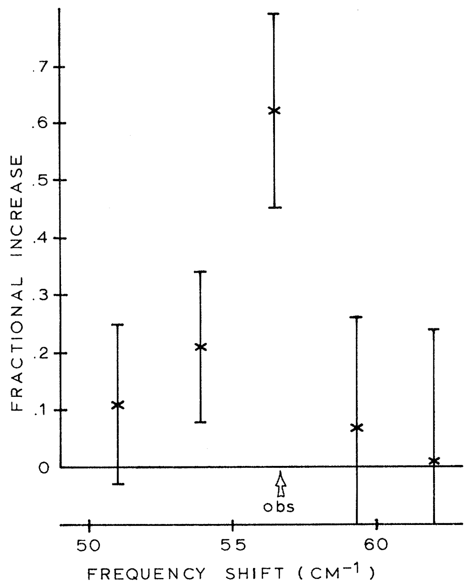

FIG. 3. The fractional increase of the satellite intensity as a function of the frequency difference between the dye lasers. The shift indicated by the arrow is that of the normally observed satellite.

lite, and the intensity of the satellite was measured with and without the dye-laser beams. By varying the wavelength of one of the dye lasers we were able to tune the frequency difference through the plasma resonance.

Results. –The results obtained are shown in Fig. 3, where we have plotted the increase in the satellite intensity as a function of the wave-number separation between the two dye beams. The error bars represent the standard deviation of the mean of at least 10 shots at each point (except for the 62-cm^{-1} shift where 5 shots were taken). It can easily be seen that there is an increase in the satellite intensity when the dye lasers are tuned to the resonance of the plasma. The slight increase on the low-frequency side may prove to be significant.

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[1]N. Kroll, A. Ron, and N. Rostoker, Phys. Rev. Lett. 13, 83 (1964).

[2]D. Montgomery, Physica (Utrecht) 31, 693 (1965).

[3]G. Weyl, Phys. Fluids 13, 1802 (1970).

[4]D. J. Bradley, G. M. Gale, M. Moore, and P. D. Smith, Phys. Lett. 26A, 378 (1968).

[5]P. W. Chan and R. Nodwell, Phys. Rev. Lett. 16, 22 (1966).

[6]B. Ahlborn, Z. Naturforch. 20a, 466 (1965).

Nonlinear Development of the Beam-Cyclotron Instability

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A theory is presented for the nonlinear evolution of the beam-cyclotron instability, together with the results of a series of computer experiments confirming the main points of the theory. We conclude that at a turbulent level which is proportional to $(\Omega_{ce}/\omega_{pe})^2$, a transition is made to the nonmagnetic ion acoustic instability. For small T_i/T_e, the system then evolves until the fluctuating fields become strong enough to stabilize by ion trapping.

In this Letter we report on the nonlinear theory and on numerical simulations of the beam-cyclotron instability.[1-4] The instability passes through three distinct phases of evolution. At first, the beam-cyclotron instability develops quasilinearly. When the turbulence reaches a prescribed (relatively low) level, anomalous wave-particle interactions smear out the individual cyclotron resonances, and the instability becomes a nonmagnetic ion acoustic instability. This ion acoustic instability then evolves quasilinearly, if $T_i \ll T_e$, until the fluctuating electric fields become large enough to trap ions, stabilizing the system. The fields are then maintained at a nearly steady level and the plasma heating rate is very much reduced.

The beam-cyclotron instability arises in plasmas when ions drift with speed v_d relative to electrons, across a magnetic field. For $v_d^2 > c_s^2 \equiv T_e/M$, ion modes can couple to electron Bernstein modes to drive an instability.[1-4] If an electron's cross-field diffusion is included in its unperturbed orbit, the dispersion relation in the electron frame, for times short compared to Ω_{ci}^{-1}, is given by

$$1-\frac{1}{2k^2\lambda_D^2}\frac{T_e}{T_i}Z'\left(\frac{\omega-kv_d}{kv_i\sqrt{2}}\right)-\frac{\omega_{pe}^2}{k^2}\int d^3v\left[1-\sum_n\frac{\omega J_n^2(k_\perp v_\perp/\Omega_{ce})}{\omega-n\Omega_{ce}+i\Delta\omega_k(v)}\right]\frac{1}{v_\perp}\frac{\partial f_e}{\partial v_\perp}=0, \qquad (1)$$

where $\Delta\omega_k(v)$ is k^2D, D being the cross-field electron diffusion coefficient. In (1), Z' is the derivative of the well-known plasma dispersion function, the J_n are Bessel functions, $v_i \equiv (T_i/M)^{1/2}$ is the ion thermal speed, λ_D is the electron Debye length, and ω_{pe}, Ω_{ce}, and Ω_{ci} are the electron plasma, electron cyclotron, and ion cyclotron frequencies, respectively. In a collisionless plasma, where cross-field diffusion is caused by turbulent fields,[5] an equation for $\Delta\omega_k(v)$ has been given by Dum and Dupree[6]:

$$\Delta\omega_k(v)=\frac{k^2}{\Omega_{ce}^2}\frac{e^2}{m^2}\sum_{k'}\sum_n|E_{k'}|^2\frac{F_n(k')(\Delta\omega_{k'}+\gamma_{k'})}{(\omega_{k'}-n\Omega_{ce})^2+(\Delta\omega_{k'}+\gamma_{k'})^2}. \qquad (2)$$

The quantity F_n is $F_n=\frac{1}{4}[J_{n-1}^2(k'v_\perp/\Omega_{ce})+J_{n+1}^2(k'v_\perp/\Omega_{ce})]$ which differs slightly from the result of Ref. 5 since our wave spectrum is not isotropic, but rather is one-dimensional. Assuming that $f_e(v_\perp)$ is Maxwellian with temperature T_e and that (for simplicity) $T_e/m \equiv v_e^2 > v_d^2$, then the integral and summation in (1) can be performed to give

$$1-\frac{1}{k^2\lambda_D^2}\left[\frac{T_e}{2T_i}Z'\left(\frac{\omega-kv_d}{kv_i\sqrt{2}}\right)-1-\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega}{kv_e}\cot\left(\pi\frac{\omega+i\Delta\omega_k}{\Omega_{ce}}\right)\right]=0. \qquad (3)$$

Notice that in Eq. (3), the magnetic field appears only in the argument of the cotangent.

Clearly, there are two regimes, $\Delta\omega_k \ll \Omega_{ce}/\pi$ and $\Delta\omega_k \gtrsim \Omega_{ce}/\pi$. We consider first the former case, where the dispersion relation reduces to that considered in Refs. 1-4. If $T_i \ll T_e$, the unstable modes cluster in narrow bands about the cyclotron harmonics, with growth rates given by[4]

$$\gamma_k^2=\left(\frac{m}{8\pi M}\right)\frac{n\Omega_{ce}^2}{(1+k^2\lambda_D^2)^{3/2}}-(\omega_k-n\Omega_{ce})^2, \qquad (4)$$

for $\gamma_k \lesssim kc_s$. In Eq. (4), $2\omega_k-n\Omega_{ce}=k[v_d-c_s(1+k^2\lambda_D^2)^{-1/2}]$.

In order to find the effect of these unstable waves on the electron and ion temperatures[7] and on the drift velocity, we use the quasilinear equations to evaluate the appropriate moments of $\partial f_e/\partial t$. To facilitate this calculation we assume that the ions are initially cold and that the electrons are Maxwellian. If most of the wave energy lies in modes with $k\lambda_D<1$, the results in the asymptotic time limit are

$$T_e(t)\approx(5e^2M^{1/2}v_d\textstyle\sum_k|\varphi_k|^2)^{2/5}, \qquad (5)$$

$$T_i(t)\approx\tfrac{2}{3}T_e^{3/2}/M^{1/2}v_d, \qquad (6)$$

$$\Delta v_d/v_d\approx -T_e(t)/Mv_d^2, \qquad (7)$$

$$d|\varphi_k|^2/dt=2\gamma_k|\varphi_k|^2. \qquad (8)$$

Since the growth rates for modes with $k\lambda_D<1$ are essentially independent of temperature, it is clear that T_e, T_i, and the square of the fluctuating potential, $\sum_k|\varphi_k|^2$, all grow exponentially with growth rates in the ratio 2:3:5.

If, on the other hand, all modes have $k\lambda_D>1$, then $\gamma_k \sim T_e^{-3/4}$ and we find that $|E_k|^2$, T_e, and T_i are all proportional to $t^{4/3}$ in the long-time limit. Thus, the heating rate is drastically reduced from the exponential heating rate for $k\lambda_D<1$. The neglect of all other nonlinear effects would therefore lead to the conclusion that the electrons heat rapidly to a temperature such that $k_{\min}\lambda_D \approx (\Omega_{ce}/\omega_{pe})v_e/v_d=1$. However, as we shall see below, other nonlinear effects do significantly influence the plasma heating.

As the electric fields grow, the turbulent collision frequency $\Delta\omega_k$ cannot be neglected in Eq. (3). To find an approximate expression for $\Delta\omega_k$, we average Eq. (2) over electron velocities, neglecting the explicit velocity dependence of $\Delta\omega_k$ on the right-hand side. Assuming that f_e is Maxwellian, the integration and summation can be performed as in the evaluation of the dispersion relation. The result is

$$\frac{\langle\Delta\omega_k\rangle}{\Omega_{ce}}\approx\left(\frac{\pi}{8}\right)^{1/2}\frac{\omega_{pe}^2}{\Omega_{ce}^2}\sum_{k'}\frac{k}{k'}\frac{kv_e}{\Omega_{ce}}\frac{|E_{k'}|^2}{4\pi n_0T_e}\times\operatorname{Im}\cot\left[\pi\frac{\omega_{k'}-i(\langle\Delta\omega_{k'}\rangle+\gamma_{k'})}{\Omega_{ce}}\right], \qquad (9)$$

where n_0 is the electron density.

When the electric fields reach such an amplitude[8] that $\langle\Delta\omega\rangle \gtrsim \Omega_{ce}/\pi$ [the second regime of Eq. (3)], the cotangents in Eqs. (3) and (9) become $-i$ and i, respectively. Thus the dispersion relation reduces to that in the absence of a magnetic field, and the instability goes over into an ion acoustic instability. This occurs at a level of turbulence such that

$$\sum_{k'} \frac{|E_{k'}|^2}{4\pi n_0 T_e(0)} \frac{k}{k'} = \left(\frac{8}{\pi^3} \frac{T_e(t)}{T_e(0)}\right)^{1/2} \frac{\Omega_{ce}^2}{\omega_{pe}^2} \frac{\Omega_{ce}}{k v_e(0)}. \quad (10)$$

The band structure of γ_k, which characterized the beam-cyclotron instabiltiy, vanishes, and the waves grow at the slower acoustic growth rate

$$\gamma_k \approx \left(\frac{\pi}{8} \frac{m}{M}\right)^{1/2} \frac{k v_D}{(1+k^2\lambda_D^2)^{3/2}}, \quad (11)$$

for $v_D \gg c_s$.

To study the subsequent evolution of the electron and ion temperatures, we have taken appropriate moments of the quasilinear diffusion equation for an unmagnetized plasma. For $k\lambda_D < 1$, the results are identical to Eqs. (5)-(8), except that the growth rate is given by Eq. (11) rather than by Eq. (4).

After a period of sustained growth at this slower rate, the plasma is expected to stabilize when the ions become trapped,[9] i.e., when

$$2e\varphi_{k_{tr}} = \tfrac{1}{2} M (v_d - \omega_{k_{tr}}/k_{tr})^2, \quad (12)$$

where k_{tr} is the wave number of the mode which is responsible for the trapping. Using the linear dispersion relation $\omega_k - k v_d = -k c_s (1 + k^2 \lambda_D^2)^{-1/2}$, and using Eq. (5) to relate $\varphi_{k_{tr}}(t)$ to $T_e(t)$, we

Table I. Each of the seven columns refers to a different numerical experiment. Rows 1-4 give basic parameters of the numerical experiments: mass ratio, electron-cyclotron to plasma frequency ratio, system length in cells, and number of particles. Rows 5-7 refer to the beam-cyclotron phase: harmonic number of the largest-amplitude mode, measured growth rate, and maximum theoretical beam cyclotron growth rate. Note that there was no observable beam-cyclotron phase in Run 1. Rows 8 and 9 give the measured level of turbulence $\Sigma |E_k|^2/4\pi n_0 T_e(0)$ at the "knee," and the value predicted by Eq. (10). Rows 10 and 11 give the measured growth rate in the acoustic phase and the maximum theoretical ion-acoustic growth rate. Rows 12-14 refer to ion trapping: $k_{tr}\lambda_D$, the measured value of T_e/Mv_d^2 at the onset of trapping, and the value predicted by Eq. (13) for the observed $k_{tr}\lambda_D$.

	Run Number						
	1	2	3	4	5	6	7
1) M/m	1836	1836	1836	1836	400	400	400
2) Ω_{ce}/ω_{pe}	0.1	0.2	0.2	0.4	0.1	0.2	0.4
3) System Size	512	1024	512	512	512	512	512
4) No. Particles	140,000	140,000	100,000	140,000	140,000	100,000	140,000
5) Harmonic No.	beam-cyclotron instability not seen	2	2	1	5	2	1
6) $\gamma_{M,cyc}/\omega_{pe}$		.0056	.0071	.017	.010	.013	.025
7) $\gamma_{Th,cyc}/\omega_{pe}$		.015	.016	.022	.020	.026	.036
8) $E^2/4\pi n_o T_e(0)\|_M$	not seen	.016	.03	.12	.002 to .004	.015	.15
9) $E^2/4\pi n_o T_e(0)\|_{Th}$	.0017 (in thermal noise)	.010	.017	.14	.0014	.014	.14
10) $\gamma_{M,sound}/\omega_{pe}$	.0033	.0017	.0012	.00074	.0071	.0056	.0014
11) $\gamma_{Th,sound}/\omega_{pe}$	.0025	.0022	.0022	.0025	.0096	.0060	.0054
12) $k_{tr}\lambda_D$	not run to saturation	1.1	0.88	1.3	0.78	0.72	1.3
13) $T_e/Mv_D^2\|_M$		.008	.012	.0056	.04	.025	.025
14) $T_e/Mv_D^2\|_{Th}$		.005	.010	.0019	.015	.019	.002

find that ion trapping occurs when the electron temperature is given by the equation

$$T_e \approx \left(\frac{5}{16}\right)^2 M v_d^{\,2}\left(1+\frac{k_{\rm tr}^{\,2}T_e}{m\omega_{pe}^{\,2}}\right)^{-4}. \tag{13}$$

To summarize, we have shown that the beam-cyclotron instability goes over to an ordinary ion acoustic instability at a level of turbulence given by Eq. (10). Stabilization will be due to the linear saturation mechanism $k_{\rm min}\lambda_{\rm D}=1$ *only* if (1) this occurs at a lower electron temperature than does trapping, Eq. (13), and (2) it occurs before the fields reach the level of Eq. (10) (since there is no smallest unstable k for the acoustic instability). Otherwise, if $T_i \ll T_e$, the electron temperature at saturation is determined by ion trapping, Eq. (13). Typically, $k_{\rm tr}\lambda_{\rm D}(t) \gtrsim 2^{-1/2}$ (waves with smaller k are less unstable); thus the fraction of ion streaming energy that goes into electron heating is $T_e/Mv_d^{\,2} \lesssim 0.02$. Finally, for the case of warm ions$-T_i/T_e \gtrsim v_d/v_e$, $v_e > v_d$$-$ion acoustic waves are stabilized by ion Landau damping.[10] In this case, Eq. (10) becomes a condition for nonlinear saturation. This conclusion is consistent with computer experiments reported in Ref. 2.

In most cross-field collisionless shock experiments,[11] Ω_{ce}/ω_{pe} is quite small, typically about $\frac{1}{70}$. Thus, from Eq. (10), a cross-field streaming instability in the shock front is expected to become an ion acoustic instability above a very low level of turbulence (or to be stabilized at this level if T_i is too large for the acoustic instability to go).

We now report on the results of computer experiments which confirm the main points of the theory. The electrostatic code used is the same as that used by Papadopoulos *et al.*,[12] and is described by Dawson, Hsi, and Shanny[13] for the case of no magnetic field. We summarize here the results of seven experiments. Mass ratios 400 and 1836 were used; in all cases except one[14] $v_d = v_e(0)$, $\lambda_{\rm D}(0) = 4$ cells, and $T_e(0) = 100T_i(0)$. Experimental parameters are summarized in Table I. Parameters were chosen to insure that a number of unstable modes with $k\lambda_{\rm D}(0) < 1$ exist.

A typical graph (Run 3) of field energy, electron and ion temperature versus time is shown in Fig. 1. Initially, the field energy and ion and electron temperatures all grow exponentially with growth rates in the ratio 5.5:3:1.7. This is in good agreement with the quasilinear prediction. Notice the drastic slowing down of the instability occuring at about $\omega_{pe}t = 800$. This "knee"

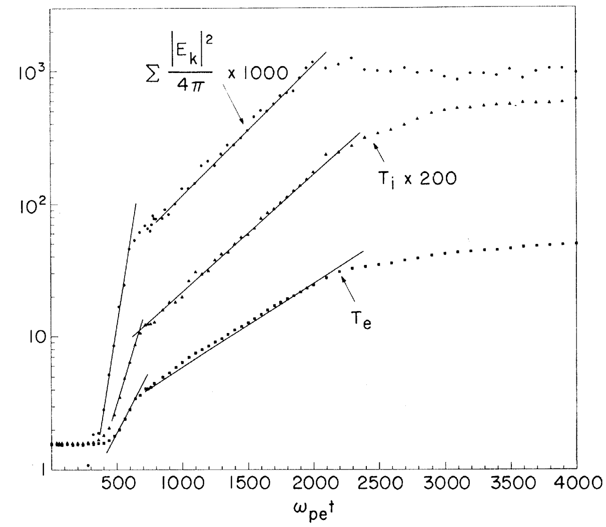

FIG. 1. Plots of electrostatic $(\Sigma E_k^{\,2}/4\pi)$, electron thermal (n_0T_e), and ion thermal (n_0T_i) energy densities, for Run 3. Energy units are arbitrary. The solid lines are drawn in to emphasize the exponential behavior during the quasilinear stages.

is a persistent feature of all runs and marks the transition from beam cyclotron to ion acoustic instability. After the "knee," the field energy and ion and electron temperatures still grow exponentially with growth rates in the ratio 3.6:3:2.1, which is in fair agreement with the quasilinear theory. At $\omega_{pe}t \approx 2200$, the instability saturates. That this saturation occurs at the onset of significant ion trapping is clearly indicated in computer print-outs of phase space.

To further demonstrate that the instability goes over to an ion acoustic instability after the "knee," we have plotted in Fig. 2 mode amplitudes at two different times, $\omega_{pe}t = 800$ and 2000. At $\omega_{pe}t = 800$, a band structure has clearly grown out of the thermal noise. However between $\omega_{pe}t = 800$ and 2000, all mode energies have increased by roughly the same factor. Thus the band structure of γ_k is smeared out for $\omega_{pe}t \gtrsim 800$.

For a time of 1600 $\omega_{pe}^{\,-1}$ after saturation, the total field energy remains remarkably constant.[15] During this time, the electron temperature increases very slowly; however the heating rate appears to be several times that expected from classical collisions. The rate is so slow and the time period so long that it is difficult to eliminate numerical effects as the source of this heating.

Finally, we present in Table I summaries of results of seven different experiments. The theoretically predicted maximum growth rate for the beam-cyclotron phase (Row 7) is in all cases

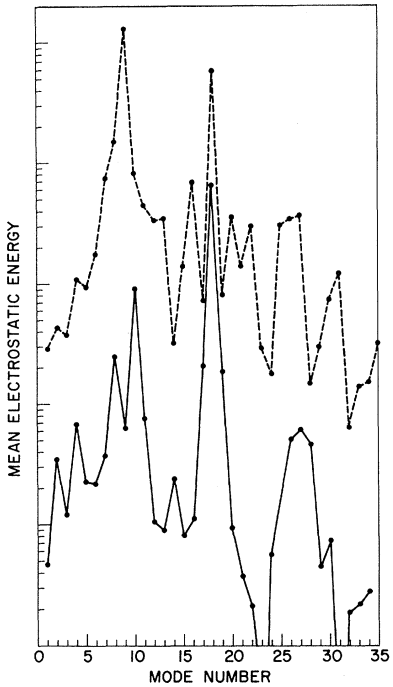

FIG. 2. Electrostatic wave energy $|E_k|^2/4\pi$ as a function of mode number, for Run 2, at times $\omega_{pe}t = 800$ (solid line) and $\omega_{pe}t = 2000$ (dashed line).

larger than the observed growth rate (Row 6) by a factor of about 1 to 2. This is to be expected since (1) it is difficult to hit exactly the fastest-growing mode, (2) other modes in the system grow at slower rates, and (3) numerical collisions are not completely negligible. There is very good agreement between experimental and theoretical values of the turbulence level at the "knee" (Rows 8 and 9), growth rate in the acoustic phase (Rows 10 and 11), and electron temperature at saturation (Rows 13 and 14). The agreement is not as good for the strong magnetic field runs.) To summarize, the results of these numerical experiments support the theoretical prediction of the transition to an ion acoustic instability and the non linear saturation by ion trapping.

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[1] H. V. Wong, Phys. Fluids 13, 757 (1970).

[2] D. W. Forslund, R. L. Morse, and C. W. Nielson, Phys. Rev. Lett. 25, 1266 (1970).

[3] S. P. Gary and J. J. Sanderson, J. Plasma Phys. 4, 739 (1970).

[4] M. Lampe, J. B. McBride, J. H. Orens, and R. N. Sudan, to be published.

[5] The quantity $\Delta\omega_k$ can be thought of as a transverse "trapping" effect due to incoherent waves in the presence of a magnetic field.

[6] C. T. Dum and T. H. Dupree, Phys. Fluids 13, 2064 (1970). This theory has also been applied to *ion*-cyclotron instabilities by C. T. Dum and R. N. Sudan, Phys. Rev. Lett. 23, 1149 (1969).

[7] By temperature we mean sloshing energy as well as random thermal energy.

[8] The theory of Dum and Dupree (Ref. 6) is strictly applicable only if $\Delta\omega_k \ll \Omega_{ce}$. Our use of the theory for the case $\Delta\omega_k = \Omega_{ce}/\pi$ gives remarkably good agreement with numerical experiments, which leads us to believe that the theory is still valid.

[9] At this amplitude it may be argued that most of the electrons would get trapped. However long-time trapping in a large-amplitude coherent wave, in the usual sense, is ruled out by the presence of even a weak magnetic field, because electrons trap and untrap at a rate faster than the electron-cyclotron frequency.

[10] T. E. Stringer, J. Nucl. Energy, Pt. C 6, 267 (1964).

[11] See Ref. 3 for a list of references on the subject of collisionless shocks.

[12] K. Papadopoulos, R. C. Davidson, J. M. Dawson, I. Haber, D. A. Hammer, N. A. Krall, and R. Shanny, to be published.

[13] J. M. Dawson, C. G. Hsi, and R. Shanny, in *Plasma Physics and Controlled Nuclear Fusion* (International Atomic Energy Agency, Vienna, Austria, 1969), Vol. I, p. 735.

[14] In Run 2 we took $v_d = 0.8v_e(0)$, $\lambda_D(0) = 5$, and $T_e(0) = 100T_i(0)$.

[15] The total field energy fluctuates somewhat more after saturation for the case $T_i \sim T_e$. See also Ref. 2.