# Perturbation Theory about a Goldstone Symmetry*

Ling-Fong Li and Heinz Pagels
*The Rockefeller University, New York, N. Y. 10021*
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We show that if a Hamiltonian symmetry is realized by Goldstone bosons then, in general, one does not expect the $S$-matrix and matrix elements of currents to be analytic functions of the symmetry-breaking parameters. This modifies the nonrenormalization theorems for vector form factors and some theorems of Dashen and Weinstein.

In order to extract the content of approximate symmetries of the hadron interactions it is usual to specify the total Hamiltonian according to

$$H=H_0+\lambda H'. \tag{1}$$

Here $H_0$ is the symmetric Hamiltonian, which for definiteness we will consider to be SU(2) $\otimes$ SU(2) symmetric. $H'$ breaks the Hamiltonian symmetry down to SU(2) with strength characterized by the parameter $\lambda$. Then if $S_{\alpha\beta}(\lambda)$ is an $S$-matrix element, one implements perturbation theory by a power series in $\lambda$:

$$S_{\alpha\beta}(\lambda)=S_{\alpha\beta}(0)+S_{\alpha\beta}'(0)\lambda+\tfrac{1}{2}S_{\alpha\beta}''(0)\lambda^2+\cdots \tag{2}$$

with $S_{\alpha\beta}(0)$ the matrix element in the absence of symmetry breaking. The necessary assumption implicit in developing this power series is that $S_{\alpha\beta}(\lambda)$ is analytic in $\lambda$ near $\lambda=0$. What we show here is that if the symmetry of $H_0$ is realized by massless Goldstone bosons, then this assumption of analyticity in $\lambda$ is probably false. We find that $S_{\alpha\beta}(\lambda)$ contains terms like $\lambda\ln\lambda$. Such $\ln\lambda$ terms can quantitatively change calculations.

To see why this is so, first consider the Hamiltonian (1) with $\lambda=0$ so that $H=H_0$ is SU(2) $\otimes$ SU(2) symmetric. If the vacuum is just SU(2) symmetric, then the Goldstone theorem[1] requires that we have massless pseudoscalar mesons, the pion triplet. Now turn on the symmetry-breaking force with $H'$ SU(2) invariant. From this explicit symmetry breaking the pions acquire a common finite mass $\mu^2\sim\lambda$. Let us consider $S_{\alpha\beta}(\mu^2)$, the transition matrix element for $\beta\to\alpha$, as a sum on all Feynman diagrams,[2] and examine this as a function of $\mu^2$. In the sum of all such diagrams there will in general be a loop with a pion. We can extract this pion and make explicit the loop integration displaying the pion propagator,

$$S_{\alpha\beta}(\mu^2)=\int\frac{d^4q}{(q^2-\mu^2)}\delta^{ab}T_{\alpha\beta}{}^{ab}(q). \tag{3}$$

Here $T_{\alpha\beta}{}^{ab}(q)$ depends on $q^2$, other invariants, and characterizes the off-shell process $\pi^a(q)+\beta\to\pi^b(q)+\alpha$. To lowest order in the pion process $T_{\alpha\beta}{}^{ab}(q)$ contains no additional pions so that in this approximation it is independent of $\mu^2$. To this approximation we have, assuming uniformity of the integral in $\mu^2$,

$$\frac{\partial S_{\alpha\beta}(\mu^2)}{\partial\mu^2}=\int\frac{d^4q}{(q^2-\mu^2)^2}\delta^{ab}T_{\alpha\beta}{}^{ab}(q).$$

Since in general $T_{\alpha\beta}{}^{ab}(0)\neq 0$,[2] we conclude that in the chiral limit $\lambda\to 0$, $\mu^2\to 0$,

$$\lim_{\mu^2\to 0}\frac{\partial S_{\alpha\beta}(\mu^2)}{\partial\mu^2}=\lim_{\mu^2\to 0}\int\frac{d^4q\,\delta^{ab}}{(q^2-\mu^2)^2}T_{\alpha\beta}{}^{ab}(q)\sim\ln\mu^2$$

because the integral diverges like $\int d^4q/q^4$. Since the pion mass $\mu^2$ is proportional to $\lambda$ we have $\lim_{\lambda\to 0}\partial S_{\alpha\beta}(\lambda)/\partial\lambda\sim\ln\lambda$ so that $S_{\alpha\beta}(\lambda)$ is not analytic in $\lambda$ near $\lambda=0$. Higher orders in the pion interactions can be expected to change the details of this result by terms like $(\ln\lambda)^2$ and even perhaps sum up. But we do not expect the feature of nonanalyticity at $\lambda=0$ to be changed.

As an explicit example consider the electromagnetic-charge form factor of the proton $F_1(q^2)$. From a once-subtracted dispersion relation one obtains for the mean charge radius

$$\tfrac{1}{6}\langle r_N^2\rangle=F_1'(0)=\frac{1}{\pi}\int_{(2\mu)^2}^{\infty}\frac{dq^2}{q^4}\operatorname{Im}F_1(q^2). \tag{4}$$

In the approximation of retaining just the two-pion intermediate state one finds

$$\operatorname{Im}F_1(q^2)=[(q^2-4\mu^2)/q^2]^{1/2}(q^2-4\mu^2)\times F_\pi^*(q^2)A(q^2)\theta(q^2-4\mu^2). \tag{5}$$

Here the first and second factors are two-body phase space and a $P$-wave factor, respectively. $F_\pi(q^2)$ is the pion form factor, and $A(q^2)$ is a $J=1$ projection of the process $\pi+\pi\to N+\bar{N}$ which is nonvanishing at $q^2=0$ and at the chiral limit, as can be seen in Born approximation in the $\sigma$ model. From (4) and (5) we find that as $\mu^2\to 0$, $F_1'(0)\sim\ln(\mu^2/m^2)$, reflecting the fact that the charge distribution is infinite in the chiral limit.

In the light of these observations we will reexamine some theorems which have assumed the validity of perturbation theory in symmetry breaking. We will assume that $H_0$ is SU(3)⊗SU(3) invariant and that this symmetry is realized by SU(3) multiplets and an octet of pseudoscalar Goldstone bosons.

Our remarks apply only to perturbation theory about the SU(3)⊗SU(3)-symmetric Hamiltonian and not to perturbations about the SU(3) symmetry of the Hamiltonian. To distinguish these possibilities we split up the SU(3)⊗SU(3)-breaking Hamiltonian:

$$\lambda H' = \beta H_1 + \gamma H_2 \tag{6}$$

with $H_1$ SU(3) invariant but not SU(3)⊗SU(3) invariant and $H_2$ violating SU(3) symmetry. If $\gamma = 0$, then $H$ is SU(3) invariant and the pseudoscalar octet is degenerate but not necessarily massless. To lowest order in the symmetry breaking their common mass $\mu^2$ is proportional to $\beta$. Since the mesons are not massless if $\gamma = 0$, one can probably do power-series expansions in $\gamma$ but not in $\lambda$. Perturbative theorems about SU(3) breaking, like the Behrends-Sirlin-Ademollo-Gatto[3] theorem, are perfectly valid (provided the dynamics of $H'$ does not require $\lambda = 0$ if $\gamma = 0$). Nonetheless it is of interest to examine these theorems if one does perturbation theory in $\lambda$ since the matrix elements of $\beta H_1$ and $\gamma H_2$ could be comparable. Perturbative theorems like those of Dashen and Weinstein,[4,5] since they assume perturbation theory in $\lambda$, are necessarily altered.

To examine the modification of the nonrenormalization theorem[3] and the correction to Gell-Mann–Okubo mass formulas, we will use the method of Furlan, Lannoy, Rossetti, and Segrè.[6] They have shown that the renormalization correction to the vector charge form factors at zero momentum transfer between states $|a\rangle$ and $|a'\rangle$ is given by $C = \sum_{\alpha'} (C_{\alpha'} - C_{\alpha''}) \delta(\vec{p}_a - \vec{p}_{a'})$, where the sum excludes the single-particle state. After explicitly extracting the factor $\gamma^2$, we will consider what remains in the chiral limit $\lambda \to 0$, understood as $\gamma \to 0$ then $\beta \to 0$.

If the external states $\langle a|$ and $|a'\rangle$ are baryon states and the intermediate state $\alpha$ is the one-baryon, one-meson state, then the correction factor in the $|\vec{p}| \to \infty$ frame is[6]

$$\lim_{|\vec{p}|\to\infty} C_{\alpha'} = \frac{\gamma^2 \pi}{2(2\pi)^3} \int_{(m+\mu)^2}^{\infty} \frac{ds}{(s-m^2)^3} \int_{\eta_1}^{\eta_2} d\eta\, \Phi^\alpha(s,\eta),$$

$$\eta_{1,2} = \frac{s+m^2}{2s}(m^2-\mu^2) \mp \frac{s-m^2}{2s}\{[s-(m-\mu)^2][s-(m+\mu)^2]\}^{1/2}, \tag{7}$$

where $m$ is the baryon mass, $\mu$ the meson mass, and $\Phi^\alpha(s,\eta)$ an invariant amplitude which in nonvanishing at the threshold $s = (m+\mu)^2$ and the chiral limit. In writing (7) we have explicitly extracted a factor $\gamma^2$ and evaluated the remainder in the SU(3) limit. As $\mu^2 \to 0$, the integral (7) diverges like $\ln(\mu^2/m^2)$, and we conclude that such states contribute renormalization effects of $O(\gamma^2 \ln\lambda)$, i.e., $F_1^B(0) = 1 + O(\gamma^2 \ln\lambda)$.

If the external states $\langle a|$ and $|a'\rangle$ are pseudoscalar-meson states the procedure is similar. In the $|\vec{p}| \to \infty$ frame the first such state allowed by parity is the three-pseudoscalar-meson state. For example, for the $f_+(t)$ form factor in $K_{l3}$ decay one concludes that $f_+(0) = 1 + O(\gamma^2 \ln\lambda)$.

In the $|\vec{p}| \to \infty$ frame one gets the correction to the quadratic mass formulas[6,7] $\Delta M^2 = 2M_N^2 + 2M_\Xi^2 - 3M_\Lambda^2 - M_\Sigma^2$, $\Delta\mu^2 = 4\mu_K^2 - 3\mu_\eta^2 - \mu_\pi^2$ in terms of just class-I states. Proceeding as before one gets for the corrections $\Delta M^2 = O(\gamma^2)$, $\Delta\mu^2 = O(\gamma^2)$ which are the usual results. From these we see that the linear mass formula for the mesons is $2\mu\Delta\mu = O(\gamma^2)$ or $\Delta\mu \sim O(\gamma^2/\mu) \sim O(\gamma^2/\sqrt{\lambda})$ which can also be gotten in the $|\vec{p}| = 0$ frame. From the point of view of chiral perturbation theory the quadratic formula for the mesons is of lower order and hence should be better. This is valid experimentally.

Next we examine the theorem of Dashen and Weinstein for $K_{l3}$ decay.[4] They conclude from chiral SU(3)⊗SU(3) perturbation theory that

$$D(0) = (\mu_K^2 - \mu_\pi^2) + O(\lambda^3), \tag{8}$$

$$D'(0) = \tfrac{1}{2}[f_K/f_\pi - f_\pi/f_K] + O(\lambda^2), \tag{9}$$

where $D(t) = \langle \pi^0(p')|\partial_\mu V_\mu^{K^+}(0)|K^+(p)\rangle$, $t = (p-p')^2$, with $V_\mu^{K^+}$ the strangeness-changing vector current. Equation (8) follows from the nonrenormalization theorem for $f_+(0)$ related to the divergence by $D(0) = (\mu_K^2 - \mu_\pi^2) f_+(0)$. Since $\mu_K^2 - \mu_\pi^2 \sim O(\gamma)$, using the result on $f_+(0)$ we conclude that (8) should be replaced by

$$D(0) = (\mu_K^2 - \mu_\pi^2) + O(\gamma^3 \ln\lambda), \tag{10}$$

For $D'(0)$ we find no conclusion like (9) can be made.[9] If one retains just the $\pi K$ intermediate state, the resulting Omnès equation for $D(t)$ can be solved exactly and implies that

$$D'(0)=[D(0)/\pi]\int_{(\mu_\pi+\mu_K)^2}^{\infty}dt\,\delta_0(t)/t^2.$$

$\delta_0(t)$ is the $S$-wave $\pi$-$K$ scattering phase shift. In the SU(3) limit $\mu_\pi=\mu_K=\mu$, $\delta_0(t)=(t-4\mu^2/t)^{1/2}A_0(t)$. In the chiral limit the $S$-wave meson-meson scattering length vanishes, and we find $A_0(t)\sim t/f^2$ near threshold. The integral diverges like $\ln\mu^2$ as $\mu^2\to 0$. Hence $D'(0)\sim O(\gamma\ln\lambda)$ which is incompatible with (9) if the first term is $O(\gamma)$. Besides the first term of (9), one finds that there are additional terms $O(\gamma)$.[4]

Finally we examine the hadronic corrections to the Goldberger-Treiman relations. The matrix element of the divergence of the axial current between nucleon states can define the off-shell $\pi N$ vertex $K_{\pi N}(t)$ by $D_A(t)=\sqrt{2}\mu_\pi^2 f_\pi K_{\pi N}(t)/\mu_\pi^2-t$, $K_{\pi N}(\mu_\pi^2)=g_{\pi N}$. The correction to the Goldberger-Treiman relation is then

$$\delta_{\pi N}=2M_N g_A-\sqrt{2}g_{\pi N}f_\pi=-\sqrt{2}\mu_\pi^2 f_\pi K_{\pi N}'(0),$$

$$K_{\pi N}'(0)=\frac{1}{\pi}\int_{(3\mu)^2}^{\infty}\frac{dt\,\mathrm{Im}K_{\pi N}(t)}{t(t-\mu_\pi^2)}, \tag{11}$$

and similarly for $\Lambda$ and $\Sigma$ decay

$$\delta_{K\Lambda}=(M_\Lambda+M_N)g_A^{(\Lambda)}-g_{K\Lambda}f_K$$
$$=-\mu_K^2 f_K K_{K\Lambda}'(0),$$
$$\delta_{K\Sigma}=(M_\Sigma+M_N)g_A^{(\Sigma)}-\sqrt{2}g_{K\Sigma}f_K$$
$$=-\mu_K^2 f_K K_{K\Sigma}'(0). \tag{12}$$

Since the explicit factor of the meson mass appearing on the right is of $O(\lambda)$, Dashen and Weinstein[5] evaluate $K'(0)$ in the chiral limit. However $K'(0)$ diverges like $\ln\mu^2$ in this limit. This follows from (11) as $\mu^2\to 0$ if one retains just the $3\pi$ state for which $\mathrm{Im}K_{\pi N}(t)=\rho_3(t,\mu)R(t,\mu^2)$ with $\rho_3$ the three-body equal-mass phase space and $R(t,\mu^2)$ nonvanishing at threshold and the chiral limit.[8] After extracting this logarithmic factor the remainder is finite in the chiral limit. By parametrizing this piece in terms of $F$ and $D$, the new sum rule is gotten by eliminating $F$ and $D$ from

$$\delta_{\pi N}=\mu_\pi^2 f_\pi\ln(3\mu_\pi/M)\sqrt{2}(F+D),$$
$$\delta_{K\Lambda}=\mu_K^2 f_K\ln(\mu_K+2\mu_\pi/M)(-\sqrt{2}/3)(3F+D),$$
$$\delta_{K\Sigma}=\mu_K^2 f_K\ln(\mu_K+2\mu_\pi/M)\sqrt{2}(D-F). \tag{13}$$

If one assumes that the $F/D$ ratio in (13) is equal to the same ratio for the meson-baryon couplings one has

$$\Delta_{\pi N}/\Delta_{K\Lambda}=(\mu_\pi^2/\mu_K^2)\ln(3\mu_\pi/M)/\ln(\mu_K+2\mu_\pi/M) \tag{14}$$

with $\Delta_{\pi N}=-\delta_{\pi N}/\sqrt{2}g_{\pi N}f_\pi=0.08\pm0.02$, $\Delta_{K\Lambda}=-\delta_{K\Lambda}/g_{K\Lambda}f_K\approx 0.32\pm0.10$. The left side is $\approx 0.25$ and the right side is $\approx 0.24$ with $M=M_N$; the ratio of the logarithms give a factor of 4.

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