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[8]T. T. Wu and C. N. Yang, Phys. Rev. 137, B708 (1965).

[9]J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, Phys. Rev. 188, 2159 (1969).

[10]The notations are the same as in Bjorken and Paschos (Ref. 5).

[11]The heuristic argument leading to Eq. (2) is due to C. N. Yang, private communication. In Ref. 9, it is essentially the same heuristic argument that leads to Eq. (37).

[12]J. G. Rutherglen, in *Proceedings of the Fourth International Symposium on Electron and Photon Interactions at High Energies, Liverpool, England, 1969*, edited by D. W. Braben and R. E. Rand (Daresbury Nuclear Physics Laboratory, Daresbury, Lancashire, England, 1970).

[13]The average multiplicity is independent of the choice of $C(Q^2,\nu)$.

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Multiplicity Distribution and Single-Particle Spectrum in the Diffractive Model*

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It is first argued that in the diffractive model the multiplicity distribution should behave as n^{-2} at high n. It is then shown how this behavior can be obtained in the statistical description of the two clusters, in which the cluster multiplicity is proportional to its mass. Finally the single-particle spectrum is calculated without any free parameters except for normalization and is shown to agree remarkably well with present data that give a cross section that is constant modulo a factor $\ln s$.

Various models[1-5] have been suggested for the description of high-energy collision processes. Among them the diffractive model[5] is successful in obtaining both the correct s dependence (e.g., the constancy of $\sigma_{\rm tot}$ modulo a factor $\ln s$) and the correct t dependence[6,7] (as exemplified by the derivation of the Wu-Yang conjecture[8]). In this paper we extend the consideration to multiparticle-production processes.

The basic assumption underlying the diffractive model for multiparticle production is that at high energies the bulk of the secondary particles can be regarded as forming two clusters, each emanating from one of the two impinging particles.[9,2] The Pomeranchukon that is exchanged is self-reproducing,[5] and is a weak branch point fixed at $j=1$.[6] Because it is fixed at $j=1$, the differential cross section $d\sigma/dt$ and the n-particle production cross sections $\sigma_n(s)$ are limiting as $s\to\infty$. In the droplet model[2] this limiting behavior is hypothesized. Here and in the following we shall use the word "limiting" as in Ref. 2, if a cross section is constant modulo a factor $\ln s$, a dependence which will be ignored.

The total cross section σ_t and average multiplicity $\langle n\rangle$ are defined as

$$\sigma_t=\sum\nolimits_n\sigma_n;\quad \langle n\rangle=\sigma_t^{-1}\sum\nolimits_n n\sigma_n. \tag{1}$$

The constancy of σ_t is an empirical fact at least up to 60 GeV/c.[10] At laboratory energies most σ_n are still increasing,[11] but in cosmic-ray experiments the preliminary results[12] are consistent with limiting σ_n. Assuming that this is borne out by more accurate experiments, the question is then how $\langle n\rangle$ can increase with $\ln s$, as suggested by the Echo Lake experiment.[13] The answer evidently resides in the n dependence of the limiting σ_n. A convergent σ_t but a divergent $\langle n\rangle$ can be achieved in (1) if the high-n behavior of σ_n is n^{-c}, $1<c\le 2$. If $\langle n\rangle$ indeed diverges asymptotically as $\ln s$, then $c=2$, since

$$\langle n\rangle\propto\int_2^{N(s)}dn\,n^{-1}\propto\ln s. \tag{2}$$

Let us now see whether this inverse-square behavior of the multiplicity distribution is compatible with data. The Echo Lake experiment[13] has data at five energy intervals. They are shown in

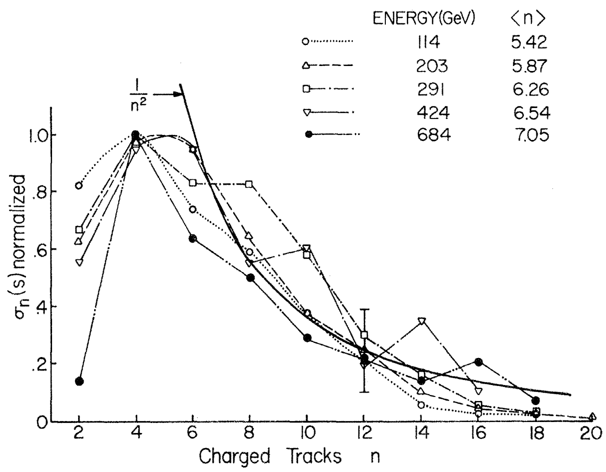

FIG. 1. Echo Lake cosmic-ray data (Ref. 13) on multiplicity distribution, normalized to unity at maxima. A typical error bar is shown at $n=12$.

Fig. 1. The curves are normalized to unit height at maxima so that the shapes at different energies can be compared. Being cosmic-ray data, the error bars are large, a typical one being shown at $n=12$. It is evident from Fig. 1 that for $n\geq 8$ the distribution is compatible with n^{-2}. Moreover, it is interesting to point out[14] that the peak of the distribution curve does not appear to shift persistently with energy in the same way that $\langle n\rangle$ does, contrary to the predictions of the multiperipheral model (MPM). To be sure, the data are too crude to distinguish between MPM and the diffractive model. We wish only to establish here that the diffractive model is a potential candidate and proceed in the following to build more structure into the model so that not only the n^{-2} behavior obtains, but also the one-particle spectrum can be determined.

The additional specification of the diffractive model that is needed is obviously about the nature of the two clusters of secondary particles. We assume that the two incident particles are excited diffractively into two massive states,[15] which subsequently decay isotropically in their respective rest frames according to some statistical distribution. Because of the known transverse-momentum distribution,[11,16] the "temperature" of the excited states should be such that pions are "evaporated" with an average energy E of about 350 MeV in the cluster rest frames. Thus the rest masses of the excited states, M_1 and M_2, should be proportional to the number of particles, n_1 and n_2, that they emit. There is some experimental evidence[17] in support of this assertion. Note that we do not assume that the multiplicity distribution of the clusters is analogous to that of the hadronic collisions. The Pomeranchukon is not like a hadron. Obviously, the model described here is more meaningful if n_1 and n_2 are not too small.

The differential cross section for n particles in the final state is

$$d\sigma_n=(4p^*s^{1/2})^{-1}|T_n|^2d\varphi_n, \qquad (3)$$

where p^* is the magnitude of the incident momentum in the c.m. frame, and

$$d\varphi_n=[\prod_{i=1}^{n}(2\pi)^{-3}(2p_i^0)^{-1}d^3p_i](2\pi)^4\delta^4(P-\sum_{i=1}^{n}p_i). \qquad (4)$$

It is straightforward to show that (4) may be rewritten in the form appropriate for two clusters of particles ($n=n_1+n_2$):

$$d\varphi_n=ds_1ds_2d\varphi_{12}d\varphi_{n_1}d\varphi_{n_2}(2\pi)^{-2} \quad (s_{1,2}=M_{1,2}^2), \qquad (5)$$

where $d\varphi_{12}$ is the phase space for the two excited states,

$$d\varphi_{12}=(2\pi)^{-2}(4K_1^0K_2^0)^{-1}d^3K_1d^3K_2 \times\delta^4(P-K_1-K_2), \qquad (6)$$

and $d\varphi_{n_1}$ and $d\varphi_{n_2}$ are as given by (4) with P replaced by K_1 and K_2, respectively. According to our assumptions of the two-cluster model, we write

$$T_n=\sum\nolimits_{n_1}T_{12}(s,s_1,s_2,t)G_{n_1}(p_i)G_{n-n_1}(p_j), \qquad (7)$$

where T_{12} is the amplitude of production of the two excited states, while G_{n_1} and G_{n_2} are their decay amplitudes. Since the Pomeranchukon is fixed at $\alpha=1$, diffractive dominance implies that $T_{12}\sim i\beta_{12}(t)(s/s_1s_2)$, ignoring the factor $\ln s$ arising from the cut nature of the Pomeranchukon. The "reduced" strength function $\beta_{12}(t)$ is well behaved at $t=0$ and is factorizable.[7] We assume for it a diffraction peak whose width broadens with the cluster masses M_α, i.e.,

$$\beta_{12}(t)=\beta_0e^{Bt}, \quad B=b(M_1^{-1}+M_2^{-1}), \qquad (8)$$

where b is a constant. This is based partly on phenomenology[18] and partly on the physical reasoning[2] that it is easier to transfer momentum if the incident particles are allowed to be excited. For $K_\parallel\gg K_\perp$ (momentum of the excited states in c.m. frame), $d\varphi_{12}\approx dt/8\pi s$. Hence, to sum up we have at large s

$$\sigma_n=\frac{\beta_0^2}{16\pi^3}\int dn_1dn_2ds_1ds_2dt \times(s_1s_2)^{-2}e^{2Bt}\Phi_1\Phi_2\delta(n_1+n_2-n), \qquad (9)$$

where

$$\Phi_\alpha(s_\alpha, n_\alpha) = \int |G_{n_\alpha}(p_i)|^2 d\varphi_{n_\alpha}.$$

Now, Φ_α is invariant so we can consider the integral in the cluster rest frame where $\vec{K}_\alpha = 0$. If the decay distribution $|G_{n_\alpha}(k_i)|^2$ is exponential[19] or Gaussian in the momenta k_i in this frame, Φ_α turns out to be sharply peaked at $s_\alpha^{1/2} \sim n_\alpha E$. In order to render the main features of the model transparent, we make here the ideal simplification (with refinements to be reported elsewhere):

$$\Phi_\alpha(s_\alpha, n_\alpha) = \gamma(n_\alpha)\delta(s_\alpha^{1/2} - n_\alpha E), \tag{10}$$

where $E \approx 350$ MeV. The normalization $\gamma(n_\alpha)$ is determined by the asymptotic behavior of the Pomeranchukon-hadron scattering amplitude, $T_{Ph}(s_\alpha)$, since $\mathrm{Im}\,T_{Ph}(s_\alpha) = \Sigma_{n_\alpha}\Phi_\alpha(s_\alpha, n_\alpha) \propto \gamma(s_\alpha^{1/2})$. We assume that the triple coupling of Pomeranchukons is negligible,[20] and that T_{Ph} is therefore dominated by the leading non-Pomeranchukon f, whose zero intercept is $\alpha_f(0) \approx \frac{1}{2}$. Thus, $\gamma(s_\alpha^{1/2}) \propto s_\alpha^{1/2}$ or equivalently $\gamma(n_\alpha) = \gamma_0 n_\alpha$. Using this in (10) and then (9) we obtain at large n

$$\sigma_n = \mathrm{const}\; n^{-2} \ln n. \tag{11}$$

The $\ln n$ factor in (11) should not be taken seriously since $\alpha_f(0) \approx \frac{1}{2}$ is only approximate. The data as shown in Fig. 1 are, of course, too crude to distinguish such a logarithmic dependence. If, however, one does take (11) literally, then $\langle n \rangle \propto (\ln s)^2$ which is also compatible with data. What we have shown in that our model can yield realistic results on σ_n and $\langle n \rangle$. The specific assumptions used are the form of $\beta_{12}(t)$ and the lack of triple Pomeranchukon coupling.

We now proceed to derive the one-particle spectrum in an inclusive reaction. We let

$$|G_{n_\alpha}(k_i)|^2 \propto \prod_i^{n_\alpha} g(k_i),$$

where $g(k_i)$ is a Gaussian distribution with invariant normalization:

$$g(k) = 2k^0(6\pi/E^2)^{3/2}\exp(-3k^2/2E^2), \tag{12}$$

$$\int g(k)(2\pi)^{-3}(2k^0)^{-1}d^3k = 1. \tag{13}$$

In computing $2p^0 d\sigma_n/d^3p$ where p is the c.m. momentum of an observed pion, we need $2p^0 d\Phi_\alpha/d^3p$, as is evident in (9). The latter in the αth cluster rest frame is clearly $(2\pi)^{-3}g(k)\Phi_\alpha$, from which (10) can trivially be recovered by integration using (13). Hence,

$$2p^0 d\sigma_n/d^3p = (2\pi)^{-3}\int[g_1'(p) + g_2'(p)]d\sigma_n, \tag{14}$$

where $g_\alpha'(p)$ is the distribution in the c.m. frame as transformed from the αth cluster rest frame. In terms of $x = 2p_\parallel s^{-1/2}$ and $\xi = 2K_\parallel s^{-1/2}$, where $K_\parallel$ is the c.m. longitudinal momentum of the clusters, we have, for the decay of the forward cluster ($\alpha = 1$), $k_\parallel/E \approx n_1 x/\xi - 1$, where $\xi^2 = (1 - n^2\epsilon^2)[1 - (n_1 - n_2)^2\epsilon^2]$, $\epsilon^2 = E^2/s$. After integrating out $d^2k_\perp$, the transformation is

$$\int_\perp g(k)(2\pi)^{-3}(2k^0)^{-1}d^3k \to (3/2\pi)^{1/2} n_1 \xi^{-1}\exp[-\tfrac{3}{2}(n_1 x/\xi - 1)^2]dx. \tag{15}$$

This indicates that the distribution in x for every fixed n_1 is peaked at $x = \xi/n_1$ with width $\Delta x \sim \xi/n_1$ and height $\sim n_1/\xi$. For n_1 and $n_2 \ll \epsilon^{-1}$, we may approximate $\xi \approx 1$; this is adequate so long as x is not in the wee-x region[3] $[x \sim O(\epsilon)]$, where the two-cluster picture is not very meaningful and phase-space roundoff occurs anyway. For simplicity we take the limit $s \to \infty$ first and then study the x dependence for all x. We obtain for $x > 0$

$$\frac{d\sigma_n}{dx} = \frac{C}{n}\int_{n_0}^{n-1}(n - n_1)^{-1}\exp[-\tfrac{3}{2}(n_1 x - 1)^2]dn_1, \tag{16}$$

where $C = (3/2\pi)^{1/2}(\beta_0\gamma_0)^2/8\pi^3 bE^4$. The minimum number n_0 in a cluster should be ≥ 2, because we aim to obtain the spectrum for unfavored particles.[2]

The invariant one-particle distribution is

$$F(x) = \textstyle\sum_n n x\, d\sigma_n/dx. \tag{17}$$

Replacing $\sum_n^N \int_{n_0}^{n-1} dn_1$ by $\int_{n_0}^{N-1} dn_1 \int_{n_1+1}^{N} dn$, where $N = s^{1/2}/m$, and integrating, we get

$$F(x) = Cx\int_{n_0}^{N-1}dn_1\exp[-\tfrac{3}{2}(n_1 x - 1)^2]\ln(N - n_1)$$
$$\simeq C(\pi/24)^{1/2}\ln s\{1 - \mathrm{erf}(\zeta(x))\}, \tag{18}$$

where $\zeta(x) = (\frac{3}{2})^{1/2}(n_0 x - 1)$. This is a limiting distribution, again modulo a factor $\ln s$. If the cut nature of the weak Pomeranchukon[5,6] had been taken into account in T_{12} in (7), an extra factor of an inverse power of $\ln s$ would have appeared in (18). Thus ignoring the multiplicative $\ln s$ factors in (18), $F(x)$ is finite at $x = 0$. The shape is independent of E or the "temperature" of the excited states. Except for C there are no free parameters, n_0 being the minimum cluster multiplicity which depends upon the type of inclusive reaction. For forward-going π^+ in $\pi^- + p \to \pi^+ +$ anything, for example, $n_0 = 3$; the corresponding $F(x)$ is shown by the solid curve in Fig. 2. For comparison, the data[21,22] at 25 GeV/c are also shown. It is claimed[22] that limiting behavior is attained already at that energy except in the wee-x region ($|x| \lesssim 0.1$). The agreement between our prediction and the data is evidently

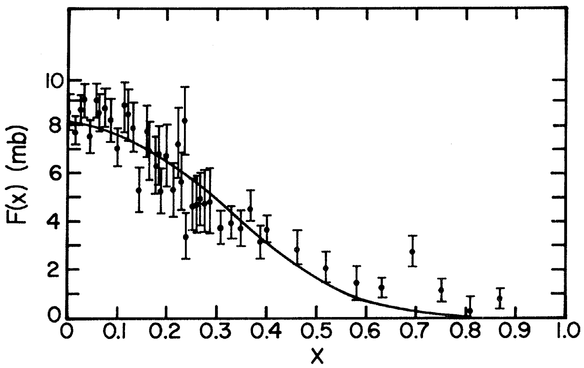

FIG. 2. Single-particle spectrum for $\pi^- + p \to \pi^+ +$ anything. Data are given in Refs. 21 and 22 for $P_{\rm beam} = 25$ GeV/c; $F(x) = x^0 d\sigma/dx$, where $x^0 = (x^2 + 0.18/s)^{1/2}$. Solid curve is the theoretical prediction at $s = \infty$ with normalization chosen to fit the data.

good.

As energy increases, the wee-x region shrinks, and we expect the agreement to become even better for small x. Our prediction in the region $0.5 < x < 1.0$ is somewhat low. This is expected because our model is inadequate in that region where only the $n_1 = 3$ cluster contributes significantly, since the peaks of $d\sigma_{n_1 n_2}/dx$ are centered at $x = n_1^{-1}$ and $-n_2^{-1}$. A cluster of such low multiplicity is not expected to decay isotropically or statistically. On the other hand, the present model implies the interesting result that, for $|x| \gtrsim 0.5$, $F(x)$ is determined only by how a particle is diffractively excited into a *minimal* cluster. This can presumably be checked using existing data such as $\pi^- + p \to (\pi^+ + \pi^- + \pi^-) + p$. Since the decay of the minimally excited pion is different from that of the proton, we expect the pion spectrum in πp collision to be asymmetric for $|x| > 0.5$.

Although the broad maximum at small $|x|$ and the general shape throughout all x do not depend on the details of the statistical description of the decay of the two excited states, the approximate slope of $F(x)$ for $|x| < 0.5$ is dependent on the minimum cluster multiplicity n_0. It is easy to see from (18) that this slope is proportional to n_0. Thus we expect that the spectrum is symmetrical about $x = 0$ for $|x| \lesssim 0.4$ if the values of n_0 for the minimal clusters of the target and of the projectile are the same. A reaction such as $K^+ + p \to \pi^- + \cdots$ would be an example if all mass differences in the minimal cluster could be ignored. On the other hand, in a reaction like $p + p \to \pi^\pm + \cdots$, the slope of the π^- spectrum should be steeper than that of π^+ since $n_0 = 3$ for π^- but $n_0 = 2$ for π^+.

The asymmetry of the forward and backward small-x distributions in the $\pi^\pm p$ and K^+p interactions[21,23,24] can also be understood in this model. The data indicate that the pions emanating from the proton cluster contribute to a spectrum with a steeper slope than those from the projectile cluster. This is merely a mass effect in this model. The minimal cluster for π is $n_0 = 3$; all three pions have the same average momentum, $\langle x \rangle \approx \frac{1}{3}$. But in $p \to p + \pi^+ + \pi^-$, although $n_0 = 3$ also, the proton being heavier takes away most of the momentum leaving each pion with $|\langle x \rangle| < \frac{1}{3}$. A rough kinematical estimate indicates that $|\langle x \rangle| \approx \frac{1}{5}$. For smaller values of $|x|$ where high-multiplicity events are important, such mass effects are less significant. Hence, the slope on the proton side is steeper than on the meson side.

In concluding we want to dispel the common notion that two-cluster production necessitates a dip in $F(x)$ at $x = 0$. This is the two-fireball model. In the diffractive-statistical model discussed here where the cluster multiplicity increases linearly with its mass, the central region is filled with high-multiplicity events resulting in a broad maximum.

I am grateful to Professor H. T. Nieh and Professor C. N. Yang for helpful discussions.

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ERRATUM

DIELECTRIC PROPERTIES OF $SrTiO_3$ AT LOW TEMPERATURES. T. Sakudo and H. Unoki [Phys. Rev. Lett. 26, 851 (1971)].

Equation (3) should read

$$G=(2C)^{-1}(T-T_c)\sum P_i^2+\tfrac{1}{2}\sum c_{ijkl}x_{ij}x_{kl}+G_\varphi+G_{\text{int}},$$

and Eq. (7) should read

$$\chi_i^{\text{eff}}=C^{-1}(T-T_c)+\Delta\chi_i.$$