This is the master equation for the evolution of the P representation in the Schrödinger picture. If, as an example, one applies Eq. (17) to the simple case

$$H=\omega(t)a^{\dagger}a, \tag{18}$$

one has

$$i\frac{\partial}{\partial t}P(\alpha^*,\alpha,t)=\omega\left[\alpha^*\frac{\partial}{\partial\alpha^*}P(\alpha^*,\alpha,t)-\alpha\frac{\partial}{\partial\alpha}P(\alpha^*,\alpha,t)\right] \tag{19}$$

which is the well-known equation[4] for the Hamiltonian described by Eq. (18).

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*NATO Fellow, on leave of absence from Fondazione Ugo Bordoni, Istituto Superiore Poste e Telecomunicazioni, Roma, Italy.

†NATO Fellow, on leave of absence from Istituto Elettrotecnico, University of Naples, Naples, Italy.

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Fourth Test of General Relativity: New Radar Result

Irwin I. Shapiro,* Michael E. Ash,† Richard P. Ingalls,‡ and William B. Smith†
Massachusetts Institute of Technology, Cambridge, Massachusetts 02319

and

Donald B. Campbell, Rolf B. Dyce, Raymond F. Jurgens,§ and Gordon H. Pettengill‖
Arecibo Observatory, Arecibo, Puerto Rico

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New radar observations yield a more stringent test of the predicted relativistic increase in echo times of radio signals sent from Earth and reflected from Mercury and Venus. These "extra" delays may be characterized by a parameter λ which is unity according to general relativity and 0.93 according to recent predictions based on a scalar-tensor theory of gravitation. We find that $\lambda=1.02$. The formal standard error is 0.02, but because of the possible presence of systematic errors we consider 0.05 to be a more reliable estimate of the uncertainty in the result.

General relativity predicts that the round-trip time delay of an electromagnetic wave is influenced by the gravitational potential along the path of the radiation. A test of this prediction involving the transmission of radar signals from Earth to either Mercury or Venus and the detection of the echoes was suggested in 1964.[1] These echoes are expected on the basis of general relativity to be retarded by solar gravity by an amount[2]

$$\Delta t\simeq(4r_0/c)\ln[(r_e+r_p+R)/(r_e+r_p-R)], \tag{1}$$

where Δt, expressed in harmonic coordinates, is the coordinate-time retardation, $r_0\simeq1.5$ km is the gravitational radius of the sun, c is the speed of light far from the sun, r_e is the Earth-sun distance, r_p is the planet-sun distance, and R is the Earth-planet distance. The quantity Δt is not an observable but is indicative of the magnitude and behavior of the measurable effect as predicted by general relativity. The operational interpretation of the effect has been discussed in detail elsewhere.[3] To test whether or not the echo time-delay data are in agreement with this theory, we may insert an *ad hoc* multiplicative parameter λ on the right side of Eq. (1) and estimate it along with the other unknown parameters that affect the data.[4]

This experiment was first performed in 1967 and yielded the result[5] $\lambda=0.09\pm0.2$ which corresponds to a value of 0.8 ± 0.4 for γ, the relevant coefficient in the generalized metric for the Schwarzschild solution.[6] Over the past three years a substantial body of consistent echo time-delay and (less important) Doppler data has acculated from radar observations of Mercury, Venus, and Mars made at the Haystack Observatory and from observations of Mercury and Venus made at the Arecibo Observatory. (The inconsistencies in the data from these two observatories, noted previously,[5] have been resolved for Venus and will be the subject of a separate publication; small differences in the Mercury data are still under investigation.) The "crucial" measurements near superior conjunctions were obtained primarily at Haystack, but only for Mercury and Venus. The Arecibo measurements were of most use in the refinement of the orbits of Earth and Venus.[7] How do these additional data improve the estimate of λ? With 24 relevant parameters[8] estimated simultaneously in a weighted least-squares analysis, we obtained

$$\lambda = 1.01_5 \pm 0.02, \tag{2}$$

or, equivalently,

$$\gamma \simeq 1.03 \pm 0.04, \tag{3}$$

where 0.02 is the formal standard error in the estimate of λ. This error reflects a uniform reduction by a factor of 0.6 of the individual measurement errors; it was made so that the root mean square of the weighted post-fit residuals for the approximately 1700 measurements would be unity. When the known topographical effects are taken into account (see below and Fig. 1), these residuals decrease appreciably, providing a further indication that the individual measurement errors originally assigned were in most cases quite conservative.

There are two main sources of systematic error that might affect this result: (i) the solar corona, and (ii) uncharted topography on the target planets. Since the accuracy of the echo-delay measurements is so much greater at inferior than at superior conjunctions, the necessity to estimate the planetary orbits *per se* introduces no important errors—either systematic or otherwise.

The solar corona and, more generally, the interplanetary plasma cause the group echo delays to be increased by an amount inversely proportional to the square of the radar signal frequency which is 430 MHz at Arecibo and 7840 MHz at Haystack. Because of the former's relatively

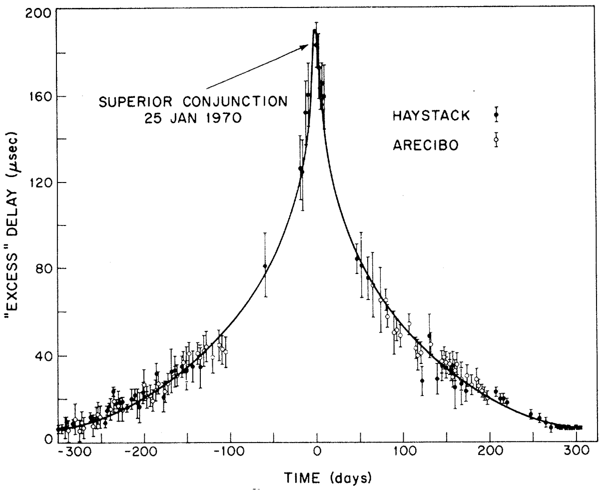

FIG. 1. Typical sample of post-fit residuals for Earth-Venus time-delay measurements, displayed relative to the "excess" delays predicted by general relativity. Corrections were made for known topographic trends on Venus. The bars represent the original estimates of the measurement standard errors. Note the dramatic increase in accuracy that was obtained with the radar-system improvements incorporated at Haystack just prior to the inferior conjunction of November 1970.

low radar frequency, the interplanetary medium affects the time-delay data noticeably. To take this plasma into account approximately, we assumed the interplanetary medium to be static with a charged-particle density varying as the inverse square of the distance from the sun. This law appears from other evidence to be an adequate representation for heliocentric distances between about 20 and 200 solar radii—a range encompassing the ray paths for all of the Arecibo radar measurements. The proportionality constant ρ, normalized to yield the electron density at the Earth's orbit, was one of the estimated parameters.[8] The result

$$\rho = 7 \pm 2 \text{ electrons/cm}^3 \tag{4}$$

represents an average condition over the period 1966-1970 spanned by the most precise Arecibo Venus data. Other more sensitive measurements yield average values between about 5 and 7 electrons/cm^3 in good agreement with our estimate which cannot be expected to be nearly so accurate in view of the limited magnitude of the effect —only about 2 μsec for Arecibo Venus observations near elongation—and the possible systematic errors that might have been introduced by planetary topography.

Inside 20 solar radii, the coronal density increases more steeply with decreasing distance to the sun than in the inverse-square model. Nevertheless, for the Haystack measurements, the two-way coronal delay most likely never exceeded 3 μsec.[9] Thus, even though the model underestimates delays near superior conjunction, the value of λ could thereby be increased spuriously by no more than a few percent.

Planetary topography affects the time delay in two ways: First, the altitude at, and in the neighborhood of, the subradar point on the target planet will directly influence the time delay; and, second, the radar scattering law,[10] which varies with aspect, will exert an indirect influence through the cross-correlation or "template-matching" technique that is used to estimate the delay to the subradar point.[5] In principle these effects can be determined once and for all and taken into account. There is no question of dynamics: The time scale for changes in surface height and in scattering law are undoubtedly long compared to the decade-length scale of these radar astronomy experiments. Although considerable progress has been made recently in charting surface-height and scattering-law variations[11] for the inner planets, uncertainties still remain that, for example, might contribute an error of up to 5 or 10 μsec to the interpretation of some of the Venus echo delays. To investigate empirically the sensitivity of our result for λ to such uncertainties, we performed a number of computer experiments in which different sets of parameters were estimated and different subsets of data deleted. In some, the parameters included low-order terms in the sectoral harmonic expansions for the effective surface heights in the equatorial regions spanned by the subradar points. For more than a dozen such computer experiments, the variations in λ in all but three instances were smaller than the formal standard error. However, the full spectrum of topographic effects cannot be investigated economically by such studies, and we therefore cannot set accurate limits on the potential contribution of systematic errors. Our best judgement is that the contributions from these sources raise the uncertainty to an equivalent standard error of about 0.05 for the estimate of λ. A sample of the post-fit Earth-Venus delay residuals, displayed relative to the "excess" delays given by Eq. (1), is shown in Fig. 1 after correction for known topographic variations.[11] When the remaining topographic uncertainties are reduced sufficiently, the analysis can be repeated, and the resultant formal standard error may then be a reliable measure of the accuracy of the λ estimate.

How does this result for λ compare with the value predicted by the Brans-Dicke scalar-tensor theory of gravitation? Based primarily on his interpretation of the Princeton solar-oblateness experiment, Dicke[12] considers the most likely value of s, the fractional contribution of the scalar field, to be 0.07 which implies a value for λ of 0.93. This prediction appears to differ significantly from our determination.

The prospects for substantial improvement of this test of general relativity seem good. Continued radar observations, especially near the inferior conjunctions of Mercury and Venus, would allow substantial gains to be made in topographic determinations. But these must be accompanied by increased measurement accuracy near superior conjunctions. A greater signal-to-noise ratio is required for this purpose than is available with Haystack alone. The new Jet Propulsion Laboratory Goldstone 210-ft-diam antenna and radar system will yield an increase in sensitivity of more than an order of magnitude, but operates at the relatively low frequency of 2388 MHz. The solar corona, therefore, becomes a serious problem, primarily because of its dynamic behavior which makes the development of an adequate parametrized model difficult at best. Pulsar data, for example, disclose variations of a factor of 3 in the integrated electron density in one day, and of 20% in 3 h, relative to the average behavior of the corona at ray-path distances of 5 to 15 solar radii.[9] However, the Goldstone system has been modified so that it can also receive radar echoes at the 7840 MHz frequency at which the Haystack transmitter operates. This bistatic "Goldstack" configuration should yield substantial improvements in delay-measurement accuracy relative to the Haystack system.[13] By alternating monostatic S-band with bistatic X-band measurements, the contribution of the solar corona can be inferred. This combination will be particularly potent with Venus as the target since its orbit relative to Earth's is known best, and since its resonance rotation[14] insures that approximately the same "face" appears at the subradar point at superior conjunction as at inferior conjunction where the topography can be well determined. The error in the estimate of λ from such measurements would probably be no greater than 1%.

Spacecraft with so-called ranging transponders may offer the best hope for improvement in the accuracy of this time-delay test. Presently operating spacecraft, unfortunately, are hindered in the achievement of higher accuracy both by nongravitational forces which affect their orbits in a manner difficult to model adequately and by the variable characteristics of the solar corona which contribute important errors to the interpretation of the signal delay for the S-band frequency used on the spacecraft.[15] Both problems may be alleviated significantly in the future: the former by "anchoring" the spacecraft to a planet, as in the planned 1971 Mariner and 1975 Viking missions to Mars, and the latter by employing dual S- and X-band transponders as in the planned 1973 Mariner Venus/Mercury and Viking missions. If time-delay data are obtained near superior conjunction with these spacecraft, the error in the estimate of λ, and consequently of γ, may be reducible to 0.1%.

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*Department of Earth and Planetary Sciences and Department of Physics.

†Lincoln Laboratory, Lexington, Mass. 02173; supported in part by the U. S. Department of the Air Force.

‡Haystack Observatory.

§Now at University of Ohio, Athens, Ohio 45701.

‖Now at the Department of Earth and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, Mass. 02139.

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[9]This result is based on the integrated coronal electron densities found from pulse time-of-arrival measurements made daily at Arecibo during the June 1969 and 1970 occultations of the Crab pulsar (C. C. Counselman and J. M. Rankin, private communication).

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