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Galerkin Approximations to Flows within Slabs, Spheres, and Cylinders

Steven A. Orszag*
National Center for Atmospheric Research,† Boulder, Colorado 80302
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This Letter introduces infinite-order accurate and efficiently implementable Galerkin (spectral) approximations to time-dependent incompressible flows within slabs, spheres, and cylinders with either rigid no-slip or free-slip boundaries. The unusual choice of Chebyshev polynomials as Galerkin expansion functions is crucial for the efficiency of the method.

Numerical simulation of time-dependent incompressible flows in slab, spherical, and cylindrical geometries is of much current interest in fluid dynamics. Applications include studies of nonlinear effects in rotating fluids, nonlinear instability, and turbulence. For flows in slabs with either periodic or free-slip boundary conditions, it has recently been shown[1-4] that Galerkin (spectral) approximations obtained using expansions in Fourier series permit substantial economies in both computer time and storage necessary to achieve reasonable standards of accuracy. It has been demonstrated[3] that, in n space dimensions, fourth-order finite-difference approximations [i.e., schemes for which the error due to using a discrete space variable is $O(\Delta x^4)$, where Δx is the grid scale] require *at least* 2^n times as many degrees of freedom to achieve reasonable accuracy as Galerkin (Fourier) approximations; on the other hand, recent improvements[2,4] in the transform methods used to implement the Galerkin equations have made computation times per time step comparable to that of finite-difference simulations involving the *same* number of independent degrees of freedom. All comparisons were made in the absence of significant time-differencing errors.[3] The advantages of the Galerkin approximations are enhanced when compared with the more commonly employed second-order finite-difference approximations. Also, if very accurate (or moderately accurate long-time) simulations are required, the Galerkin approximations offer the advantage of giving infinite-order approximations [cf. Sect. (1)] to infinitely differentiable flows.

In this Letter, we describe infinite-order accurate and *efficiently implementable* Galerkin approximations to flows within slabs with rigid boundary conditions. We remark on the extension to flows in cylindrical and spherical geometries below. Instead of expanding in series of trigonometric or Chandrasekhar-Reid functions[5] (that exhibit Gibbs's phenomenon in some velocity derivative at rigid boundaries), we expand the flows in Chebyshev polynomials.

(1) *Some properties of expansions in orthogonal polynomials.*—Consider a function $v(x)$ having derivatives of all orders for $|x|<1$ and one-sided

derivatives of all orders at $x=\pm 1$ [though $v(x)$ need not be analytic]; the properties stated below must be weakened accordingly if $v(x)$ has only a finite number of derivatives. Let $T_n(x)$ denote the nth-degree Chebyshev polynomial of the first kind. It is possible to expand $v(x)$ as[6]

$$v(x)=\sum_{n=0}^{\infty} a_n T_n(x), \tag{1}$$

$$a_n = h_n \int_{-1}^{1} v(x) T_n(x) w(x) dx,$$

where $h_n = 2/\pi c_n$, $c_0 = 2$, $c_n = 1$ $(n \neq 0)$, and $w(x) = (1-x^2)^{-1/2}$. It may be shown[7] that

$$a_n = O(1/n^p) \quad (n \to \infty) \tag{2}$$

for any finite number p. Since $|T_n(x)| \leq 1$ for $|x| \leq 1$, it follows from (2) that the remainder after N terms of (1) decreases faster than any power of $1/N$, uniformly for $|x| \leq 1$. In this sense, the orthogonal expansions are infinite order. Since

$$d^q T_n(x)/dx^q = O(n^{2q}) \quad (n \to \infty) \tag{3}$$

for any integer q, it follows from the known convergence[6] of (1) to $v(x)$ for $|x| \leq 1$ and from uniform-convergence arguments that the Chebyshev expansion may be differentiated termwise any number of times for $|x| \leq 1$. In other words, Chebyshev expansions do not exhibit Gibbs's phenomenon in any velocity derivative at rigid boundaries if $v(x)$ is infinitely differentiable. Exactly the same results hold for expansions in Legendre polynomials[2] and wide classes of other orthogonal polynomials.

(2) *Galerkin (Chebyshev) approximations to incompressible flows.* — The Navier-Stokes equations for incompressible flow are

$$\partial v_\alpha/\partial t + \partial(v_\beta v_\alpha)/\partial x_\beta = -\partial p/\partial x_\alpha + \nu \nabla^2 v_\alpha, \tag{4}$$

$$\partial v_\beta/\partial x_\beta = 0, \tag{5}$$

where $\vec{v}(\vec{x}, t)$ is the three-dimensional velocity field, p is the pressure (normalized by the density), and ν is the kinematic viscosity; repeated Greek subscripts are summed over their range, $\alpha = 1, 2, 3$. To be definite, we choose periodic boundary conditions with period 2π in x_1 and x_2 and no-slip conditions ($\vec{v}=0$) for $x_3 = \pm 1$, although this choice is not crucial. Let a set of (nonorthogonal) polynomials be defined by

$$q_{2n}(x) = T_{2n}(x) - T_0(x),$$

$$q_{2n+1}(x) = T_{2n+1}(x) - T_1(x) \tag{6}$$

for $n \geq 1$, so that $q_n(\pm 1) = 0$. We seek an approximate solution to (4) of the form

$$v_\alpha(\vec{x}, t) = \sum u_\alpha(k_1, k_2, n, t) \exp[i(k_1 x_1 + k_2 x_2)] q_n(x_3), \tag{7}$$

where the sum $\sum$ extends over all intergers k_1, k_2, and n satisfying the inequalities $|k_1| < K_1$, $|k_2| < K_2$, and $2 \leq n \leq N$; K_1, K_2, and N are integer cutoffs. The expansion (7) automatically satisfies the imposed boundary conditions although it should not be expected that exact solutions to (4) are representable in the truncated form (7). Similarly, the pressure is expanded in a series of the form (7) with coefficients $p(k_1, k_2, n, t)$.

Approximate equations to determine the time evolution of $u_\alpha(k_1, k_2, n, t)$ are found as follows: Suppose that the Navier-Stokes equations are written in formal operator form as $N\vec{v} = 0$, and let the expansion functions in (7) be denoted by $\psi^{k_1 k_2 n}$. The Galerkin equations are $(Nv_\alpha, \psi^{k_1 k_2 n}) = 0$, where (7) is formally substituted for $\vec{v}$ in the Navier-Stokes equations and (f,g) is the inner product over $[0, 2\pi] \times [0, 2\pi] \times [-1, 1]$ with weight $w(x_3)$. The equations found by this procedure are most simply expressed by introducing the notation

$$u_\alpha(k_1, k_2, 0, t) = -\sum_{m=1}^{M} u_\alpha(k_1, k_2, 2m, t), \quad u_\alpha(k_1, k_2, 1, t) = -\sum_{m=1}^{M} u_\alpha(k_1, k_2, 2m+1, t), \tag{8}$$

where we choose $N = 2M+1$; hence $q_n(x_3)$ is replaced by $T_n(x_3)$ in (7) if the sum on n ranges on $0 \leq n \leq N$. We also define

$$\bar{u}_\alpha(k_1, k_2, n, t) = c_n u_\alpha(k_1, k_2, |n|, t) \tag{9}$$

for $|n| \leq N$, where c_n is defined after (1).

After some considerable algebra, the Galerkin equations become

$$c_n\frac{\partial}{\partial t}u_\alpha(k_1,k_2,n,t)=b_\alpha^{\ n}(k_1,k_2,t)-i(\delta_{\alpha,1}+\delta_{\alpha,2})c_nk_\alpha p(k_1,k_2,n,t)-2\delta_{\alpha,3}\sum_{\substack{m=n+1\\ m+n\equiv 1\ (\mathrm{mod}\ 2)}}^{N} mp(k_1,k_2,m,t)$$
$$-\tfrac{1}{2}i\sum_{j=2}^{2}k_j{\sum}'\bar{u}_j(p_1,p_2,m,t)\bar{u}_\alpha(k_1-p_1,k_2-p_2,n-m,t)$$
$$-{\sum}''(m_1+m_2)\bar{u}_3(p_1,p_2,m_1,t)\bar{u}_\alpha(k_1-p_1,k_2-p_2,m_2,t)-\nu c_n(k_1^{\ 2}+k_2^{\ 2})u_\alpha(k_1,k_2,n,t)$$
$$+\nu\sum_{\substack{m=n+2\\ m\equiv n\ (\mathrm{mod}\ 2)}}^{N} m(m^2-n^2)u_\alpha(k_1,k_2,m,t), \tag{10}$$

$$\sum_{j=1}^{2}ik_jc_nu_j(k_1,k_2,n,t)+2\sum_{\substack{m=n+1\\ m+n\equiv 1\ (\mathrm{mod}\ 2)}}^{N} mu_3(k_1,k_2,m,t)=0, \tag{11}$$

where $\delta_{i,j}=1$ if $i=j$; $\delta_{i,j}=0$ if $i\neq j$; $\sum'$ indicates a sum over all p_1, p_2, m satisfying $-K_r<p_r, k_r-p_r<K_r$ $(r=1,2)$, $-N\leq m, n-m\leq N$; $\sum''$ indicates a sum over all p_1, p_2, m_1, m_2, satisfying $-K_r<p_r, k_r-p_r<K_r$ $(r=1,2)$, $-N\leq m_1, m_2\leq N$, $m_1+m_2\geq n+1$, $m_1+m_2+n\equiv 1$ (mod 2); and $b_\alpha^{\ n}(k_1,k_2,t)=b_\alpha^{\ 0}(k_1,k_2,t)$ if $n\equiv 0$ (mod 2), and $b_\alpha^{\ n}(k_1,k_2,t)=b_\alpha^{\ 1}(k_1,k_2,t)$ if $n\equiv 1$ (mod 2). Equations (10) and (11) hold for $|k_r|<K_r$ $(r=1,2)$, $0\leq n\leq N$. The quantities $b_\alpha^{\ 0}(k_1,k_2,t)$ and $b_\alpha^{\ 1}(k_1,k_2,t)$ are determined in terms of $\vec{u}$ and p by the constraints (8) that follow from $\vec{v}=0$ for $x_3=\pm1$, while $p(k_1,k_2,n,t)$ is determined by the constraints (11). It follows from (8) and (11) that the Galerkin approximation (7) satisfies $\partial v_3/\partial x_3=0$ exactly at $x_3=\pm1$, as required by (5).

The relatively simple convolution form of (10) follows from the relations[6]

$$2T_nT_m=T_{n+m}+T_{|n-m|},\quad T_n{}'=nU_{n-1}, \tag{12}$$

$$\sum_{m=0}^{n}T_{2m}=\tfrac{1}{2}+\tfrac{1}{2}U_{2n},\quad \sum_{m=0}^{n}T_{2m+1}=\tfrac{1}{2}U_{2n+1}, \tag{13}$$

where U_n is the nth-degree Chebyshev polynomial of the second kind.

It is easy to see that the terms $\sum'$ and $\sum''$ in (10) require essentially no more work than if periodic boundary conditions were applied in all three space directions. First, note that $\sum''$ is a simple accumulation of sums of the form $\sum'$. Second, transform methods[4] reduce evaluation of all the required $\sum'$ ($\sum''$) sums in (10) for $\alpha=1,2,3$ to eighteen real or conjugate-symmetric discrete Fourier transforms on $2K_1\times 2K_2\times N$ points. Each of these transforms may be evaluated by the fast Fourier-transform algorithm[8] in order $K_1K_2\times N\log_2(K_1K_2N)$ operations. It may be shown that evaluation of the other terms in (10) and evaluation of $\vec{b}^0$, $\vec{b}^1$, and p by the constraints (8) and (11) requires only an additional $O(K_1K_2N)$ operations. Consequently, numerical integration of (8)-(11) requires little more computer time than integration of the equations with fully periodic boundary conditions and the same number of independent degrees of freedom.

A similar application of Galerkin's procedure, using Chebyshev polynomials as expansion functions in radius for flows in spherical and cylindrical geometries (including shells), gives equations of convolution type with only minor complications. The crucial facts here are (12), (13), and the recurrence formula $x^{-1}(T_{n+1}+T_{n-1})=2T_n$, which shows that division by x is readily accomplished within Chebyshev series. For flows in spherical geometry, it is necessary to expand the angular dependence in a surface-harmonic expansion; in cylindrical geometry, Fourier expansions are required. Transform methods[9,2] also apply here to speed the evaluation of the Galerkin equations. These Galerkin equations for flows within spheres and cylinders encounter no numerical difficulty at polar axes, *i.e.*, there are no mapping singularities or stability problems due to convergence of the mesh near the axis. It is also possible to use Chebyshev expansions efficiently with "stretched" coordinate systems that give finer resolution within, say, boundary layers.

(3) *Some numerical results.*—A simple comparison between the Galerkin (Chebyshev) and finite-difference methods is obtained for the one-dimensional wave equation

$$\partial v(x,t)/\partial t+\partial v(x,t)/\partial x=0,\quad v(-1,t)=f(t), \tag{14}$$

for $|x|\leq 1$ with $v(x,0)$ given. The Galerkin (Chebyshev) approximation to (14), gotten by techniques similar to those of Sect. (2), is

$$c_n da_n/dt = (-1)^n b(t) - 2 \sum_{\substack{m=n+1 \\ m+n\equiv 1 \pmod 2}}^{N} m a_m \qquad (0 \le n \le N), \tag{15}$$

where the Galerkin approximation is

$$v(x,t) = \sum_{n=0}^{N} a_n(t) T_n(x), \tag{16}$$

and where $b(t)$ is determined from the constraint $v(-1, t) = f(t)$.

It is possible to prove rigorously that the approximation obtained by (15) is an infinite-order approximation to the exact $v(x,t)$ (provided the latter is infinitely differentiable). First, it is shown that the column vector $\vec{\delta}$, whose components δ_n ($n = 0, \cdots, N$) are the differences between the exact and Galerkin Chebyshev coefficients, satisfies a linear inhomogeneous equation of the form $d\vec{\delta}/dt + L\vec{\delta}(t) = \vec{h}(t)$, where L is a constant $(N+1)\times(N+1)$ matrix and $\vec{h}(t)$ depends only on the exact Chebyshev coefficients for $n > N$. Next, it is shown that the eigenvalues of L have non-negative real parts, so that convenient error bounds on the Galerkin approximation are gotten in terms of bounds on $\vec{h}(t)$. The results of Sect. (1) are used to show that, as $N \to \infty$, $\vec{h}(t)$ tends to zero faster than any power of $1/N$, so the same is true of the error $\vec{\delta}(t)$. The convergence proof just given extends to quite a wide variety of problems.[10]

Accuracy of simulation of the solution to (14) with $f(t) = \sin(M\pi t)$ is an effective test of phase errors in numerical schemes.[3] With this choice of $f(t)$, the solution to (14) for $t \ge 2$ is exactly $v(x,t) = \sin[M\pi(t-x-1)]$ for $|x| \le 1$, independently of $v(x, 0)$. Since the exact Chebyshev coefficients of this $v(x,t)$ are expressible in terms of Bessel functions as $a_n = (2/c_n)\sin[(t-1)M\pi \frac{1}{2} n\pi] J_n(M\pi)$ for $t \ge 2$, it follows that the exact coefficients decrease rapidly with n for $n \gtrsim M\pi$. Therefore, it should be expected that (15) gives accurate results for $N \gtrsim M\pi$. Since there are M complete waves in the interval $[-1, 1]$, it follows that at least π degrees of freedom per wavelength are required for accurate simulations.

Equation (15) was solved numerically to check this latter prediction. Notice that the calculation of the right-hand side of (15) requires only $O(N)$ operations per time step. Define E as the rms error in $v(x,t)$, $|x| \le 1$, at $t = 5$. The numerical results are that for $N = 27$, $M = 8$ [$(N+1)/M = 3.5$ degrees of freedom per wavelength], $E \approx 1.2 \times 10^{-1}$, while for $N = 31$ [$(N+1)/M = 4$], $E \approx 7.1 \times 10^{-3}$, and for $N = 35$ [$(N+1)/M = 4.5$], $E \approx 2.7 \times 10^{-4}$. Clearly, the error decreases rapidly with $1/N$. Analogous quantitative results have been found for several linear boundary-layer problems.[2]

On the other hand, a centered fourth-order finite-difference scheme with careful application of the boundary conditions (exact values at all upstream points and stable fourth-order one-sided differences downstream) gives the rms error $E \approx 5.0 \times 10^{-2}$ with forty grid points and $M = 4$ (ten grid points per wavelength), and $E \approx 3.5 \times 10^{-3}$ with eighty grid points and $M = 4$. Also, a centered second-order scheme gives $E \approx 9 \times 10^{-3}$ with 120 grid points and $M = 2$. It may be shown that these rms errors are roughly proportional to M for fixed numbers of grid points per wavelength. More detailed theoretical and numerical comparisons between the Galerkin approximations introduced here and finite-difference approximations will be given in a later paper.

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*Alfred P. Sloan Foundation Research Fellow on leave from Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Mass.

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