Theory of Nonuniform Gases (Cambridge U. Press, Cambridge, England, 1952); or J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, *Molecular Theory of Gases and Liquids* (Wiley, New York, 1954).

[9]See for example N. Bloembergen, W. H. Lowdermilk, M. Matsuoka, and C. S. Wang, Phys. Rev. A 3, 404 (1971). The explicit form of the coupling constants has also been obtained independently by N. A. Clark, thesis, Massachusetts Institute of Technology, 1970 (unpublished); Bergé, Calmettes, DuBois, and Laj (Ref. 3) obtained equivalent expressions for liquid mixtures.

[10]M. Kohler, Ann. Phys. (Leipzig) 39, 209 (1941); also see Refs. 1 and 2.

[11]N. Bloembergen, W. H. Lowdermilk, and C. S. Wang, Phys. Rev. Lett. 25, 1476 (1970).

Anomalous Resistivity Due to Electrostatic Turbulence

S. Peter Gary
Department of Physics, College of William and Mary, Williamsburg, Virginia 23185

and

J. W. M. Paul
Culham Laboratory, Abingdon, Berkshire, England

(Received 22 March 1971)

We calculate a general expression for anomalous resistivity due to weak electrostatic turbulence in a plasma. In the case of ion acoustic turbulence, it is shown to reduce to the heuristic result of Sagdeev. Applications to perpendicular collisionless shock experiments are discussed.

The heating of electrons by superthermal fluctuations in a plasma has been observed in several experiments.[1,2] This turbulent heating may be described in terms of an anomalous resistivity η^* which can be orders of magnitude larger than the resistivity due to electron-ion collisions.

To calculate η^*, we consider a stochastic model in which energy is transferred from the waves to the particles by small-angle, random scatterings of the electrons by the fluctuations. We assume a homogeneous plasma and a given spectrum of turbulence. (The inclusion of self-consistent fields is a much more difficult problem and will not be considered here.) Then the time development of the ensemble-averaged electron distribution function

$$F(\vec{v},t)=\langle f(\vec{r},\vec{v},t)\rangle$$

is given by a Fokker-Planck equation[3-5]

$$\frac{DF(\vec{v},t)}{Dt}=\frac{\partial}{\partial v_\mu}\left[D_{\mu\nu}\frac{\partial F(\vec{r},t)}{\partial v_\nu}\right], \quad (1)$$

where D/Dt represents the time derivative along the zeroth-order trajectory of an electron. If there are no fluctuating magnetic fields,[4]

$$D_{\mu\nu}=(e^2/m^2)\langle\delta E_\mu(\vec{r},t)\int_{-\infty}^{t}\delta E_\nu(\vec{r}',t')dt'\rangle, \quad (2)$$

where δE_α is the αth component of the fluctuating electric field and the t' integration is over a zeroth-order trajectory $[\vec{r}'=\vec{r}(t'),\ \vec{v}'=\vec{v}(t')]$.

We consider only electrostatic waves so that

$$\delta\vec{E}(\vec{r},t)=-\nabla\varphi(\vec{r},t), \quad (3)$$

and we work in terms of the Fourier transform of the potential-potential correlation function,

$$S(\vec{k},\omega)=\int d^3\xi\int d\tau\,\exp[-i(\vec{k}\cdot\vec{\xi}-\omega\tau)]R(\vec{\xi},\tau), \quad (4)$$

where

$$R(\vec{\xi},\tau)\equiv\langle\varphi(\vec{r},t)\varphi(\vec{r}+\vec{\xi},t+\tau)\rangle. \quad (5)$$

Then

$$D_{\mu\nu}=[e^2/(2\pi)^4m^2]\int d^3k\int d\omega\,k_\mu k_\nu S(\vec{k},\omega)\times\int_{-\infty}^{t}dt'\exp\{i[\vec{k}\cdot(\vec{r}'-\vec{r})-\omega(t'-t)]\}. \quad (6)$$

For a plasma in which there are uniform electric magnetic fields $\vec{E}_0$ and $\vec{B}_0$,

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\vec{v}\cdot\frac{\partial}{\partial\vec{r}}-\frac{e}{m}\left(\vec{E}_0+\frac{\vec{v}\times\vec{B}_0}{c}\right)\cdot\frac{\partial}{\partial\vec{v}}. \quad (7)$$

We now assume a steady-state situation in which the force exerted on the electrons by $\vec{E}_0$ is balanced by the "friction" force due to particle scattering by waves. Then from Eq. (1),

$$\frac{en_0}{m}\vec{E}_0+\frac{en_0}{mc}(\vec{v}_0\times\vec{B}_0)=-\int d^3v\,\overleftrightarrow{D}\cdot\frac{\partial F(\vec{v})}{\partial\vec{v}}, \quad (8)$$

where $\vec{v}_0=n_0^{-1}\int d^3v\,\vec{v}F(\vec{v})=v_0\hat{e}_3$ is the electron drift. Then, by analogy with magnetohydrodynamics, we define a scalar resistivity[6]

$$\vec{E}_0 + \frac{\vec{v}_0 \times \vec{B}_0}{c} = \eta^* \vec{J} = -\eta^* n_0 e \vec{v}_0 \tag{9}$$

so that

$$\eta^* = (m/e^2 n_0^2 v_0) \int d^3v\, D_{3\mu} \partial F(\vec{v})/\partial v_\mu. \tag{10}$$

We can relate this to an expression which occurs in the solution of the linear Vlasov equation. The linear dispersion relation for electrostatic waves in a Vlasov plasma is

$$1 + \sum_j K_j(\vec{k}, \omega) = 0, \tag{11}$$

where the sum is over species and, by means of integration over zeroth-order orbits,[7]

$$K_j(\vec{k}, \omega) = -(i\omega_j^2/n_0 k^2) \int d^3v\, \vec{k}\cdot(\partial f_j^{(0)}/\partial \vec{v}) \int_{-\infty}^{t} dt' \,\exp\{i[\vec{k}\cdot(\vec{r}' - \vec{r}) - \omega(t'-t)]\}, \tag{12}$$

where $\omega_j^2 = 4\pi n_0 e_j^2/mj$. Then, via Eqs. (6) and (10) and Eq. (B2) of Ref. 4, there is, to lowest order,

$$\eta^* = \frac{-2(2\pi)^{-4}}{m n_0 v_0 \omega_e^2} \int_{k_3>0} d^3k\, k_3 k^2 \int d\omega\, S(\vec{k}, \omega)\, \mathrm{Im}[K_e(\vec{k}, \omega)]. \tag{13}$$

This is the main result of this Letter. Its main limitation is that it is valid only for time scales short compared to the change in F. However, let us assume the following picture of the plasma: The flow of energy from $\vec{E}_0$ into the drift velocity, then to the enhanced fluctuations, and finally to the heating of the electrons proceeds at a constant rate. In this case, the time rate of change of $\langle f \rangle$ should be of the order of the rate of change of the macroscopic parameters, such as temperature. This is sufficiently long to give a meaningful result for η^*. We now discuss applications of Eq. (13).

Sagdeev effective collision frequency. —We consider turbulence which develops from ion acoustic waves made unstable by an electron drift, $\vec{v}_0$, relative to the ions.[8] We assume $T_e \gg T_i$ and $\omega/k < v_0 \ll v_e$, the electron thermal velocity. Our first approximation is

$$S(\vec{k}, \omega) = S(\vec{k})\delta(\omega - \omega_r(\vec{k})), \tag{14}$$

where $\omega_r(\vec{k})$ is the real part of the complex frequency which satisfies the linear dispersion relation, Eq. (11). The width of the frequency part of the spectrum, $\Delta\omega$, may be neglected if $\Delta\omega/k \ll v_e$; since (as experimental results show[2]) $\Delta\omega/k \lesssim c_s = (T_e/m_i)^{1/2}$, this is valid for ion acoustic turbulence. Experimental results[2] also show that the turbulence is anisotropic and strongest in the direction of the drift velocity. To represent this kind of behavior, we assume that

$$S(\vec{k}) = S(k)\mathcal{S}(\theta), \tag{15}$$

where $\mathcal{S}(0) = 1$ and $\mathcal{S}(\theta)$ decreases to zero as θ, the azimuthal angle measured from $\vec{v}_0$, goes to ψ, the half-angle of the cone of turbulence.

We now further specialize to the case in which $\vec{E}_0$ and $\vec{v}_0$ are parallel to $\vec{B}_0$. For $k_\parallel/k_e \gtrsim 2\Omega_e/\omega_e$ ($\Omega_e = eB_0/mc$), the linear dispersion relation reduces to the $\vec{B}_0 = 0$ result, and for Maxwellian electrons.

$$\mathrm{Im}[K_e(\vec{k}, \omega)] = \frac{k_e^2}{k^2}\pi^{1/2}\left(\frac{\omega - \vec{k}\cdot\vec{v}_0}{\sqrt{2}kv_e}\right)\exp\left[-\left(\frac{\omega - \vec{k}\cdot\vec{v}_0}{\sqrt{2}kv_e}\right)^2\right], \tag{16}$$

where $k_e^2 = 4\pi n_0 e^2/T_e$. The above three equations in Eq. (13) yield

$$\eta^* \simeq \frac{(2\pi)^{1/2}(1 - c_s/v_0)\Theta_2}{(2\pi)^4 n_0 T_e v_e} \int dk\, k^3 S(k), \tag{17}$$

where

$$\Theta_n \equiv 2\pi \int_0^{\pi/2} d\theta \sin\theta \cos^n\theta\, \mathcal{S}(\theta). \tag{18}$$

Here Θ_0 is an approximate measure of the solid angle subtended by the turbulence, and if $\psi \ll 1$, $\Theta_2 \simeq \Theta_0$.

At present, there is no generally accepted derivation of $S(k)$. The dominant mechanism for determining it has been proposed to be nonlinear ion Landau damping,[9] nonlinear electron Landau damping,[10] and wave-wave coupling.[11] Since there is now experimental evidence[1,2] that the Kadomtsev spectrum, as calculated via nonlinear ion Landau damping, exhibits the correct wave number behavior, we use this with the normalization due to Paul, Daughney, and Holmes[12]:

$$S(k) = \frac{(2\pi)^3}{7\psi^2}\frac{v_0}{v_e}\frac{T_e}{T_i}\frac{T_e^2}{4\pi e^2}\frac{\ln(k_e/k)}{k^3}. \tag{19}$$

Thus, for $v_0 \gg c_s$,

$$\eta^* \simeq (4\pi/\omega_e^2)(\Theta_2/28\sqrt{2}\pi^{3/2}\psi^2)v_0 k_e T_e/T_i. \qquad (20)$$

Experimental results[2] suggest $\psi \sim 1$ rad, so $\Theta_2/28\sqrt{2}\pi^{3/2}\psi^2 \sim 10^{-2}$ and, defining an effective collision frequency $\nu^* \equiv \omega_e^2\eta^*/4\pi$, we have

$$\nu^*/\omega_i \simeq 10^{-2}(v_0/c_s)(T_e/T_i), \qquad (21)$$

the well-known heuristic result of Sagdeev.[13] This ν^* can be substantially larger than the electron-ion collision frequency and therefore seems likely to explain anomalous electron heating observed in experiments.[1]

Perpendicular shocks. —There is evidence that turbulence associated with plasma shock waves propagating perpendicular to an applied magnetic field has properties which are independent of $\vec{B}_0$. In particular, the Culham group has shown[2] that the phase velocity of the turbulence is consistent with c_s and scales as ω_i, not as Ω_e. In addition, the wave-number spectrum appears to have the Kadomtsev k dependence [Eq. (19)] for $k \lesssim k_e$. Moreover, the Sagdeev effective collision frequency has provided order-of-magnitude agreement with the electron heating observed by Paul and co-workers[3] and by Dippel, Höthker, and Hintz.[14]

A full explanation of why the turbulence seems so $\vec{B}_0$ independent must await a complete nonlinear theory which includes magnetic field effects. At present, however, we propose the following plausibility argument as to why the Sagdeev ν^* may be valid for the perpendicular shock when $T_e \gg T_i$.

Consider a model in which the turbulence is driven by an $\vec{E}\times\vec{B}$ electron drift and the ions are not magnetized. In this case, for the parameters of the Culham experiment, the linear dispersion relation reduces to the usual expression for zero magnetic field, excepting a small range of angles about the direction perpendicular to $\vec{B}_0$.[15]

Although the growth rates in the perpendicular direction are substantially larger than those off the perpendicular, the inclusion of ∇B effects[15] tends to reduce the former. And nonlinear processes should act to distribute the wave energy randomly among the available modes of propagation, that is, away from the perpendicular direction. So we do not expect that $S(\vec{k})$ for $\vec{k}\perp\vec{B}_0$ will be substantially larger than $S(\vec{k})$ for the off-perpendicular directions.

On the other hand, Gary and Biskamp[16] have predicted that for the Culham shock the turbulence should propagate at large angles (~1 rad) to the perpendicular. Although this prediction has not yet been tested experimentally, we will assume it is correct. Then it is clear that the principal contribution to the k integration in Eq. (17) comes from off-perpendicular directions, for which $K_e(\vec{k}, \omega)$ is given by Eq. (16). In addition, the off-perpendicular waves should likewise dominate $S(\vec{k})$, so that the Kadomtsev spectrum, Eq. (19), should be valid. Thus evaluation of η^* proceeds as above, and the Sagdeev result obtains. As shown in Ref. 15, the ∇B contributions to K_e are negligible in the off-perpendicular, $\beta_e \ll 1$ case.

The ν^* can be expressed in terms of the electrostatic energy density of the turbulence, $W = \langle\vec{E}\cdot\vec{E}\rangle/8\pi$. Defining an average wave number $\bar{k}$ by

$$W/\bar{k} = \tfrac{1}{2}\Theta_0(2\pi)^{-5}\int dk\, k^3 S(k), \qquad (22)$$

Eq. (17) implies that

$$\frac{\nu^*}{\omega_i} \simeq (2\pi)^{1/2}\langle\cos^2\theta\rangle\frac{k_e}{\bar{k}}\left(\frac{m_i}{m_e}\right)^{1/2}\frac{W}{n_0 T_e}, \qquad (23)$$

where $\langle\cos^2\theta\rangle \equiv \Theta_2/\Theta_0$ is the average of $\cos^2\theta$ over the θ dependence of the turbulence. Thus we have an approximate expression for ν^* in a $T_e \gg T_i$ perpendicular shock, expressed in terms of four experimentally observable quantities. In the Culham shock, if Ref. 16 is correct, $\langle\cos^2\theta\rangle \sim \frac{1}{2}$, so in order to obtain the experimental result $\nu^* \sim \frac{1}{2}\omega_i$,

$$W/n_0 T_e \sim \tfrac{1}{2}(\bar{k}/k_e)(m_e/m_i)^{1/2}. \qquad (24)$$

These considerations do not necessarily apply for shocks with $T_e \sim T_i$ (e.g., the Garching experiments of Keilhacker and Steuer, Ref. 2). Because ion Landau damping is strong, the unstable waves (first discussed by Lashmore-Davies[17]) are confined to a relatively narrow range of angles about the perpendicular to $\vec{B}_0$.[16] If this range is so narrow that only the $k_\parallel = 0$ part of K_e is important in Eq. (13), the ∇B drift must be considered and Eq. (7) of Ref. 15 must be used for K_e. In this case it is likely that ν^* will depend on B_0.

However, if nonlinear effects are sufficiently strong to broaden $S(\vec{k})$ beyond about 3° to the perpendicular, the calculation of ν^* may proceed as above, since K_e is not affected by the increased ion damping. Thus, for $T_e \sim T_i$, if the angle of turbulence is sufficiently wide, we can expect a result like Eq. (23), in which ν^* is independent of the magnetic field.

We thank M. Keilhacker, H. Bloomberg, and F. R. Crownfield, Jr., for valuable discussions.

[1]B. A. Demidov, N. I. Elagin, and S. D. Fanchenko, Dokl. Akad. Nauk SSSR 174, 327 (1967) [Sov. Phys. Dokl. 12, 467 (1967)]; D. Dimock and E. Mazzucato, Phys. Rev. Lett. 20, 713 (1968); S. M. Hamberger and

J. Jancarik, Phys. Rev. Lett. 25, 999 (1970).

[2]J. W. M. Paul, C. C. Daughney, and L. S. Holmes, Nature 223, 822 (1969); C. C. Daughney, L. S. Holmes, and J. W. M. Paul, Phys. Rev. Lett. 25, 497 (1970); M. Keilhacker and K.-H. Steuer, Phys. Rev. Lett. 26, 694 (1971).

[3]W. B. Thompson and J. Hubbard, Rev. Mod. Phys. 32, 714 (1960); A. G. Sitenko, *Electromagnetic Fluctuations in Plasma* (Academic, New York, 1967), Chap. 9; P. A. Sturrock, Phys. Rev. 141, 186 (1966).

[4]D. E. Hall and P. A. Sturrock, Phys. Fluids 10, 2630 (1967).

[5]W. M. Manheimer and T. H. Dupree, Phys. Fluids 11, 2709 (1968); W. M. Manheimer, Phys. Fluids 12, 901 (1969).

[6]T. J. M. Boyd and J. J. Sanderson, *Plasma Dynamics* (Barnes & Noble, New York, 1969), Chap. 3.

[7]T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill, New York, 1962), Chap. 8.

[8]E. A. Jackson, Phys. Fluids 3, 786 (1960); T. E. Stringer, Plasma Phys. 6, 267 (1964).

[9]B. B. Kadomtsev, *Plasma Turbulence* (Academic, London, 1965), Sect. IV.2(a).

[10]M. L. Sloan and W. E. Drummond, Phys. Fluids 13, 2554 (1970).

[11]V. N. Tsytovich, to be published.

[12]J. W. M. Paul, C. C. Daughney, and L. S. Holmes, European Space Research Orgainzation Report No. SP-51, 1969 (unpublished).

[13]R. Z. Sagdeev, Proc. Symp. Appl. Math. 18, 281 (1967).

[14]K. H. Dippel, K. Höthker, and E. Hintz, in *Proceedings of the Fourth European Conference on Controlled Fusion and Plasma Physics, Rome, 1970* (Comitato Nazionale per l'Energia Nucleare, Ufficio Edizioni Scientifiche, Rome, Italy).

[15]S. P. Gary and J. J. Sanderson, J. Plasma Phys. 4, 739 (1970); S. P. Gary, J. Plasma Phys. 4, 753 (1970).

[16]S. P. Gary and D. Biskamp, to be published.

[17]C. N. Lashmore-Davies, J. Phys. A: Proc. Phys. Soc., London 3, L40 (1970); D. W. Forslund, R. L. Morse, and C. W. Nielson, Phys. Rev. Lett. 25, 1266 (1970).

Galerkin Approximations to Flows within Slabs, Spheres, and Cylinders

Steven A. Orszag*
National Center for Atmospheric Research,† Boulder, Colorado 80302
(Received 22 February 1971)

This Letter introduces infinite-order accurate and efficiently implementable Galerkin (spectral) approximations to time-dependent incompressible flows within slabs, spheres, and cylinders with either rigid no-slip or free-slip boundaries. The unusual choice of Chebyshev polynomials as Galerkin expansion functions is crucial for the efficiency of the method.

Numerical simulation of time-dependent incompressible flows in slab, spherical, and cylindrical geometries is of much current interest in fluid dynamics. Applications include studies of nonlinear effects in rotating fluids, nonlinear instability, and turbulence. For flows in slabs with either periodic or free-slip boundary conditions, it has recently been shown[1-4] that Galerkin (spectral) approximations obtained using expansions in Fourier series permit substantial economies in both computer time and storage necessary to achieve reasonable standards of accuracy. It has been demonstrated[3] that, in n space dimensions, fourth-order finite-difference approximations [i.e., schemes for which the error due to using a discrete space variable is $O(\Delta x^4)$, where Δx is the grid scale] require *at least* 2^n times as many degrees of freedom to achieve reasonable accuracy as Galerkin (Fourier) approximations; on the other hand, recent improvements[2,4] in the transform methods used to implement the Galerkin equations have made computation times per time step comparable to that of finite-difference simulations involving the *same* number of independent degrees of freedom. All comparisons were made in the absence of significant time-differencing errors.[3] The advantages of the Galerkin approximations are enhanced when compared with the more commonly employed second-order finite-difference approximations. Also, if very accurate (or moderately accurate long-time) simulations are required, the Galerkin approximations offer the advantage of giving infinite-order approximations [cf. Sect. (1)] to infinitely differentiable flows.

In this Letter, we describe infinite-order accurate and *efficiently implementable* Galerkin approximations to flows within slabs with rigid boundary conditions. We remark on the extension to flows in cylindrical and spherical geometries below. Instead of expanding in series of trigonometric or Chandrasekhar-Reid functions[5] (that exhibit Gibbs's phenomenon in some velocity derivative at rigid boundaries), we expand the flows in Chebyshev polynomials.

(1) *Some properties of expansions in orthogonal polynomials.*—Consider a function $v(x)$ having derivatives of all orders for $|x| < 1$ and one-sided