Criterion for the Instability of a Uniformly Rotating Configuration in General Relativity*

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(Received 22 March 1971)

Uniformly rotating configurations in general relativity are considered, and a condition is obtained that they can be quasistatically deformed without violating any of the requirements for equilibrium. This condition extends, into the domain of the rotating stars, the criterion for the onset of dynamical instability (via a neutral mode of oscillation) that occurs by radial pulsations in nonrotating stars.

It is now well known that in general relativity spherically symmetric configurations that are stable in the Newtonian theory can become dynamically unstable by a radial mode of oscillation if their radii are less than some determinate values.[1-3] It is this general relativistic instability that is responsible, for example, for the existence of a lower limit to the periods of pulsations of white dwarfs and for their exclusion as an interpretation of pulsars.[4-7]

It is also known[8-10] that in the Newtonian theory, rotation has a stabilizing effect on the spherically symmetric mode of dynamical instability in the absence of rotation. If the destabilizing effect of general relativity and the Newtonian stabilizing effect of rotation are considered, each in their lowest orders—$O(c^{-2})$ in general relativity and $O(\Omega^2)$ in rotation (c is the velocity of light and Ω is the angular velocity of rotation, assumed uniform)—then it is a simple matter to write down the result of their combined effects on stability.[9-11] But so far the problem has not been investigated in the exact framework of general relativity, and the problem is clearly important in the broader context of the collapse of rotating stars.

In this Letter we shall show how, by considering infinitesimal quasistatic deformations of uniformly rotating configurations, we can obtain a sufficient condition for the onset of instability in the exact framework of general relativity.

The description of the stationary state.—The equations governing the equilibrium of uniformly rotating axisymmetric configurations have recently been written down by several authors.[12-14] The form of the metric, which involves no loss of generality and which is convenient for our present purposes, is

$$ds^2 = -e^{2\nu}dt^2 + \varpi^2 e^{2\beta-2\nu}(d\varphi-\omega dt)^2 + e^{2\mu}(d\varpi^2+dZ^2), \qquad (1)$$

where φ is the azimuthal angle about the axis of symmetry Z; ϖ is the horizontal coordinate distance from the Z axis; and ν, μ, β, and ω are four potentials that depend only on ϖ and Z. (Units are used in which the velocity of light c and the constant of gravitation G are set equal to 1.) The field equations derived from the foregoing metric and appropriate for a perfect fluid described in terms of the energy-momentum tensor

$$T^{ij} = (\epsilon + p)u^i u^j + p g^{ij} \qquad (2)$$

have been written down by Bardeen.[14] In Eq. (2) ϵ is the energy density, p is the pressure, g^{ij} is the metric, and u^i is the four-velocity.

We shall be concerned here with quasistatic deformations of stationary uniformly rotating configurations described in terms of Eqs. (1) and (2) in order to ascertain when such deformations can be effected without violating any of the requirements for equilibrium. These conditions will be the same for the onset of instability via a neutral mode of oscillation.[15]

The description of the method.—To consider quasistatic deformations about equilibrium, it is convenient to have equations that formally govern *time-dependent* departures of the various quantities from their equilibrium values. For this purpose, we shall consider the field equations derived from a metric of exactly the same form as (1) but with the potentials ν, μ, β, and ω and the thermodynamic variables ϵ and p allowed to be functions of time as well. This choice of the time dependence is not the most general that is compatible with the basic requirement only of continued axisymmetry. But its consideration will enable us to pass conveniently to the limit of infinitely slow variations and quasistatic deformations of the equilibrium solutions [derived from Eqs. (1) and (2)] without time dependence. The procedure outlined is justified since the quasistatic deformation which we wish to consider is one which will carry one equilibrium configuration into another described by the *same form* for the metric.

Therefore, while we shall formally consider the field equations derived from a time-dependent metric of the form (1), it is to be clearly understood that our present interest in them is only in the limit of infinitely slow variations.

The generalization of Bardeen's equations, which follow from Eqs. (1) and (2) when ϵ, p, ν, μ, β, and ω are allowed to be functions of time as well, can be readily written down; and the equation governing the fluid motions may be taken to be

$$u^j u_{\alpha;j} = -(p_{,\alpha} + u_\alpha u^j p_{,j})/(\epsilon + p). \tag{3}$$

In Eq. (3) (and in the sequel) Latin indices run from 0 to 3 (with 0 and 1 referring to t and φ, respectively) and Greek indices take only the values 2 and 3 (corresponding to ϖ and Z). Summation over the respective ranges of repeated indices is to be understood; also commas will be used to indicate ordinary partial differentiation with respect to the coordinate specified, while semicolons will be used to indicate corresponding covariant differentiation.

Equation (3) must be considered together with the equations

$$[N(-g)^{1/2} u^j]_{,j} = 0 \tag{4}$$

and

$$u^j[u_1(\epsilon + p)/N]_{,j} = 0 \tag{5}$$

that ensure the conservation of the baryon number N and of the angular momentum (per baryon). In Eq. (5) u_1 is the covariant component of the four-velocity in the φ direction.

The variational equations. – Starting then from an initial configuration that is stationary and is uniformly rotating with an angular velocity Ω ($= d\varphi/dt$), we can readily write down the equations that govern deformations preserving the axisymmetry of the configuration by linearizing the time-dependent field equations as well as Eqs. (3), (4), and (5).

We shall describe the motions that will ensue in the ϖ and the Z directions by a Lagrangian displacement of the form

$$\xi^\alpha(\varpi, Z)e^{i\sigma t}, \tag{6}$$

where σ is a characteristic-value parameter to be determined. (Our principal interest is in the conditions for the existence of nontrivial proper solution ξ^α belonging to a null characteristic value.) We shall also let ΔQ and δQ denote the Lagrangian and the Eulerian changes, respectively, in any quantity Q caused by the displacement ξ^α.

The linearized version of the baryon conservation equation (4) is

$$\frac{\Delta N}{N} = -\frac{v\delta v}{1-v^2} + \delta(\nu - \beta - 2\mu) - \frac{1}{u^0(-g)^{1/2}}[u^0(-g)^{1/2}\xi^\alpha]_{,\alpha}, \tag{7}$$

where

$$v = \varpi e^{\beta - 2\nu}(\Omega - \omega) \text{ and } u^0 = e^{-\nu}(1-v^2)^{-1/2}. \tag{8}$$

From the vanishing of the covariant divergence of T^{ij} it follows that

$$\Delta\epsilon = (\epsilon + p)\Delta N/N \text{ and } \Delta p = \gamma p \Delta N/N, \tag{9}$$

where γ is an appropriate thermodynamically defined "adiabatic exponent." Making use of the foregoing relations, we find that the linearization of Eq. (5) gives

$$\frac{\delta v}{v(1-v^2)} = -\frac{\gamma p}{\epsilon + p}\frac{\Delta N}{N} - \delta(\beta - \nu) - \xi^\alpha(\ln u_1)_{,\alpha}, \tag{10}$$

where

$$u_1 = \varpi e^{\beta - \nu} v(1-v^2)^{-1/2}. \tag{11}$$

The corresponding Lagrangian change in Ω follows from Eqs. (8) and (10). We find

$$\Delta(\Omega - \omega) = -(\Omega - \omega)[\delta(\beta - 2\nu) - \delta v/v] - \xi^\alpha \omega_{,\alpha}. \tag{12}$$

And finally the linearized version of the equation of motion, Eq. (5), is

$$-\sigma^2(\epsilon + p)(u^0)^2 e^{2\mu}(-g)^{1/2}\xi^\alpha = -(-g)^{1/2}u^0(\gamma p \Delta N/u^0 N)_{,\alpha} + (\epsilon + p)(-g)^{1/2}(\Delta u^0/u^0)_{,\alpha}$$
$$+ (\epsilon + p)(-g)^{1/2}(\Delta N/N)(\ln u^0)_{,\alpha} - (\epsilon + p)u^0 u_1(-g)^{1/2}(\Delta\Omega)_{,\alpha}. \tag{13}$$

Besides Eqs. (7), (9), (10), (12), and (13), we also have the linearized versions of the field equations to determine the Eulerian changes $\delta\nu$, $\delta\mu$, $\delta\beta$, and $\delta\omega$ in the potentials in terms of δp, $\delta\epsilon$, and δv given by Eqs. (7), (9), and (10). We find, in particular, that the $(1,\alpha)$ and the $(0,\alpha)$ components of the linearized field equations admit immediate integration and give

$$16\pi\frac{\epsilon+p}{1-v^2}(\Omega-\omega)e^{2\mu}\xi^\alpha=\delta\omega_{,\alpha}+(3\delta\beta-4\delta\nu)\omega_{,\alpha}, \tag{14}$$

and

$$8\pi(\epsilon+p)/(1-v^2)(-g)^{1/2}\xi^\alpha=-(\bar{\omega}e^\beta\delta\eta)_{,\alpha}+(\bar{\omega}e^\beta)_{,\alpha}\delta\mu-\bar{\omega}e^\beta\delta\mu_{,\alpha}+2\bar{\omega}e^\beta\delta\eta\nu_{,\alpha}, \tag{15}$$

where

$$\eta=\beta-\nu. \tag{16}$$

While the relations (14) and (15) have been deduced from the $(1,\alpha)$ and the $(0,\alpha)$ components of the field equations (which vanish identically in the time-independent case), one should, in principle, be able to deduce them from the linearized versions of the *time-independent* equations. Thus, once having the relations (14) and (15), it is easy to verify that they satisfy the linearized versions of the time-independent equations

$$R^{01}=8\pi T^{01} \text{ and } R^{00}+\tfrac{1}{2}R=8\pi T^{00}, \tag{17}$$

if appropriate use is made of the conservation equations (7) and (10).

Returning to Eq. (13), we now observe that this equation, together with the usual boundary conditions on δp, $\delta\epsilon$, and the Eulerian changes in the other field variables, provides a characteristic-value problem for σ^2. It can now be shown by a somewhat lengthy analysis [in which the relations (14) and (15) play essential roles] that the characteristic-value problem is self-adjoint and that the following expression provides a variational base for determining σ^2:

$$\begin{aligned}-\sigma^2\int[(\epsilon+p)(u^0)^2e^{2\mu}|\vec{\xi}|^2-e^{-2\nu}(\delta\mu^2+2\delta\mu\,\delta\eta)/4\pi](-g)^{1/2}d^3x\\
=\int\{-(-g)^{1/2}\gamma p[1+\gamma p v^2/(\epsilon+p)](\Delta N/N)^2-(-g)^{1/2}\xi^\alpha p_{,\alpha}\xi^\beta[\ln(\epsilon+p)]_{,\beta}+(-g)^{1/2}(\xi^\alpha p_{,\alpha})^2/(\epsilon+p)\\
+2\xi^\alpha[p_{,\alpha}-\gamma p(-g)^{1/2}v^2(\ln u_1)_{,\alpha}]\Delta N/N-(\epsilon+p)v^2(-g)^{1/2}[\xi^\alpha(\ln u_1)_{,\alpha}]^2\\
-16(\epsilon+p)^2(-g)^{1/2}e^{2\mu}|\vec{\xi}|^2v^2/(1-v^2)-2(-g)^{1/2}(\delta p\,\delta v)v/(1-v^2)+4(-g)^{1/2}(\epsilon+p)u^0u_1\delta\eta\xi^\alpha\omega_{,\alpha}\\
-(-g)^{1/2}[(\epsilon+p)(1-v^2)\delta\eta^2+4(\epsilon-p)\delta\mu\,\delta\eta+2\delta\epsilon\,\delta\eta-4p\,\delta\mu^2]\\
+(\bar{\omega}e^\beta/4\pi)[\delta\eta_{,\alpha}\delta\eta_{,\alpha}-2\delta\eta\delta\mu_{,\alpha\alpha}-\bar{\omega}^2e^{2\beta-4\nu}(\omega_{,\alpha})^2\delta\eta^2]\}d^3x.\end{aligned} \tag{18}$$

Having obtained Eq. (18), we return to our main objective of finding a condition for the onset of instability via a neutral mode of oscillation, i.e., for the occurrence of a neutral state that is marginally stable. For such marginally stable states $\sigma=0$ and the required condition is obtained by setting the right-hand side of Eq. (18) equal to zero. In particular, the vanishing of the right-hand side of Eq. (18) for some chosen trial functions (that satisfy the boundary conditions of the problem and in case the integral on the left-hand side is positive-definite) will provide a sufficient condition for the onset of instability.

In the case of slow rotation, the right-hand side of Eq. (18) can be reduced, as in the Newtonian theory, to give an explicit formula for the change in the criterion for stability due to rotation, that will require only quadratures over the altered distribution [to $O(\Omega^2)$] of the various quantities in the stationary configuration and the proper solutions belonging to the radial modes of oscillation of the nonrotating spherical configuration. This reduced form of Eq. (18) can be used to determine the stability of the slowly rotating models for neutron stars that have been constructed by Hartle and Thorne.[16,17] And again, in a suitably modified form Eq. (18) can also be used to ascertain the stability of the uniformly rotating disks that have been constructed by Bardeen and Wagoner.[18]

In conclusion, it should be emphasized once again that by restricting ourselves to neutral modes ($\sigma=0$) we have avoided any reference to gravitational radiation. Gravitational radiation will undoubtedly occur when $\sigma\neq 0$; in such cases one should be careful about the boundary conditions at infinity; but these considerations are beyond the scope of this Letter.

*Work supported in part by the National Science Foundation under Contract No. GP-15973 with the University of Chicago.

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Reaction $K_L{}^0 p \to K_S{}^0 p$ from 1.3 to 8.0 GeV/c*

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Total and differential cross sections are presented for the reaction $K_L{}^0 p \to K_S{}^0 p$ from 1.3 to 8.0 GeV/c as measured in an exposure of the Stanford Linear Accelerator Center 40-in. hydrogen bubble chamber to a neutral beam. The forward points of $d\sigma(K_L{}^0 p \to K_S{}^0 p)/dt$ together with K^+n and K^-n total cross sections are used to determine the intercept of the effective Regge trajectory, $\alpha(0) = 0.47 \pm 0.09$, and the regeneration phase $\varphi_f = -43° \pm 8°$.

We present experimental result on the reaction

$$K_L{}^0 p \to K_S{}^0 p \qquad (1)$$

covering the momentum interval from 1.3 to 8.0 GeV/c. Previous investigations of Reaction (1) have been reported by Firestone *et al.*[1] in a hydrogen bubble chamber experiment and by Darriulat *et al.*[2] in a transmission regeneration experiment.

In the t channel, the reaction must proceed through exchange of neutral mesons having natural spin and parity and odd charge conjugation. The only known candidates are the members of the vector nonet (ρ, ω, and φ). As pointed out by Gilman,[3] ω exchange is expected to dominate over ρ in the forward direction, and φ exchange is expected to be negligible because of the experimentally small $\varphi \bar{N} N$ coupling. The s and t behavior of the cross section and the phase of the forward amplitude are therefore powerful tools in understanding the properties of ω exchange. A more complete analysis of these exchanges is given in the following Letter.[4]

The results presented for Reaction (1) are based on an analysis of 200 000 photographs from a total exposure of 800 000 photographs of $K_L{}^0 p$ interactions in the Stanford Linear Accelerator Center (SLAC) 40-in. hydrogen bubble chamber. The details of the beam and the $K_L{}^0$ momentum spectrum are given elsewhere.[5] We have checked carefully for possible systematic uncertainties in our determination of the $K_L{}^0$ momentum spectrum and conclude that they are negligible compared to the statistical errors of our data sample. The events were found in a scan of the one-prong-plus-vee topology, measured on conventional film plane machines, and reconstructed and fitted with the TVGP-SQUAW computer programs. The sample consists of 571 events in the momentum interval 1.3 to 8.0 GeV/c of which less than 1% are kinematically ambiguous with other hypotheses.[6] Corrections have been applied for scanning inefficiencies due to the K^0 lifetime and steeply dipping protons.