SCALE INVARIANCE AND CURRENT COMMUTATORS NEAR THE LIGHT CONE

Y. Frishman*
Weizmann Institute of Science, Department of Physics, Rehovot, Israel
(Received 17 June 1970)

Consequences of the assumption that current commutators have power-type singularities near the light cone are derived. An extension of Wilson's short-distance expansion of commutators is suggested for almost lightlike distances. Consequences for power-type singularities are derived for processes involving currents. Especially, generalized scaling laws are obtained for inelastic electron-nucleon scattering. The connection with Regge behavior is discussed. An application to electron-positron annihilation into hadrons is also made.

Inelastic lepton-hadron scattering stimulated many theoretical works.[1-11] Especially, Bjorken's conjecture[1] about the scaling behavior of inelastic electron-hadron scattering is supported by experimental data.[12] This behavior was also combined with Regge asymptotics[2] and related to the singularity structure of the matrix elements near the light cone.[7,9,10] It is the purpose of this paper to investigate the structure of commutators between currents when their space-time distance approaches the light cone. A generalization of Wilson's short-distance expansion of commutators[13] to nearly lightlike distances is proposed. This brings us naturally to power-type singularities of commutators near the light cone. Consequences are derived for inelastic electron-nucleon scattering. Generalized scaling laws are obtained for this process. In particular, a generalized scaling law applies to the $I=1$ part in the imaginary part of forward Compton scattering. We then investigate the connection with Regge behavior and consider the possibility that the leading singularity near the light cone gives also the leading Regge contribution as well as the leading term in Bjorken's scaling limit. Finally, we get a bound on the possible decrease of the electron-positron annihilation cross section from electron-proton inelastic scattering data.

The fact that light-cone singularities are important in determining certain limits of matrix elements of current commutators had been pointed out by several authors.[7,9,10] For example, consider the imaginary part of forward scattering of off-mass-shell photons of four-momentum q by protons,

$$W_{\lambda\sigma}(q,p)=\int d^4x\, e^{iqx}\langle p|[J_\lambda(x),J_\sigma(0)]|p\rangle, \qquad (1)$$

where $J_\lambda(x)$ is the electromagnetic current, p is the four-momentum of the proton ($p^2=M^2$), and a spin averaging is performed on the right-hand side. $W_{\lambda\sigma}$ for $q^2<0$ (spacelike) is related to the electron-proton scattering cross section by

$$W_{\lambda\sigma}(q,p)=\left(-g_{\lambda\sigma}+\frac{q_\lambda q_\sigma}{q^2}\right)W_1(q^2,\nu)+\frac{1}{M^2}\left(p_\lambda-\frac{p\cdot q}{q^2}q_\lambda\right)\left(p_\sigma-\frac{p\cdot q}{q^2}q_\sigma\right)W_2(q^2,\nu) \qquad (2)$$

$$\frac{d^2\sigma}{d\Omega dE'}=\frac{\alpha^2}{4E^2\sin^4\theta/2}[W_2(q^2,\nu)\cos^2\theta/2+2W_1(q^2,\nu)\sin^2\theta/2], \qquad (3)$$

where $M\nu=q\cdot p$, E and E' are the energies of the incoming and outgoing electrons, θ the scattering angle in the laboratory frame, and $\alpha=e^2/4\pi$ is the fine-structure constant. [$(1-\nu^2/q^2)W_2\geqslant W_1\geqslant 0$ for $q^2\leqslant Q$. $\tilde{W}_1$ and $\tilde{W}_2$, defined by $W_2=(-q^2)\tilde{W}_2$ and $W_1=(-q^2)\tilde{W}_1+\nu^2\tilde{W}_2$, have no extra zeros as $q^2\to 0$.] Let us now look at the Bjorken scaling limit, namely, $q^2\to-\infty$, $\nu\to\infty$, with $\omega=-q^2/2q\cdot p$ fixed. Taking the target proton at rest, this can be achieved by

$$q=\nu(1,0,0,(1+2M\omega/\nu)^{1/2}), \qquad (4)$$

with $\nu\to\infty$. Hence,

$$e^{iqx}\approx e^{i\nu(x_0-x_3)}e^{-iM\omega x_3}, \qquad (5)$$

which implies that most of the contribution comes from $|x_0-x_3|\leqslant 1/\nu$, $|x_3|\lesssim 1/M\omega$ and thus

$$x^2=[(x_0-x_3)(x_0+x_3)-(x_1^2+x_2^2)]$$

$$\lesssim[(1/M\omega\nu)-(x_1^2+x_2^2)].$$

Since $x^2 \geq 0$ only contribute to (1), we have

$$x^2 \lesssim (1/M\omega\nu) \underset{\nu\to\infty}{\longrightarrow} 0$$

and the behavior near the light cone dominates. In applying these considerations we assume that the most rapid variation of the commutator is near the light cone. Similar considerations apply to the Regge limit $\nu \to \infty$, $q^2 = -\sigma^2$ fixed. Here, choosing $q = \nu(1, 0, 0, (1+\sigma^2/\nu^2)^{1/2})$, we get $|x_0 - x_3| \lesssim 1/\nu$, $|x_3| \lesssim 2\nu/\sigma^2$, and therefore $x^2 \lesssim 1/\sigma^2$. We therefore approach nearer and nearer to the light cone the higher the value of σ^2. Following Ref. 7 we call the region for which σ^2 is large enough so that the light-cone singularities dominate the deep Regge limit. We assume here that the regular parts of the commutator inside the light cone are controlled by bound mass values, and therefore the Bjorken scaling limit and the deep Regge limit can be reached. However, in order for the deep Regge limit to be dominated by the light-cone singularities, we have to assume that the parts of the commutator which vary less in x^2, and therefore are more damped in σ^2, are also less leading in ν or at most have the same behavior in ν once σ^2 is large enough. This extra assumption is not needed for the Bjorken scaling limit, since now $\sigma^2 \to \infty$ and there is a damping in the σ^2 variable. We shall make this assumption, and so make contact between the scaling laws and the Regge limit. Our aim, therefore, is to get the structure of the commutator $[J_\mu(x), J_\nu(0)]$ near $(x-y)^2 = 0$. Let us first mention the suggestion of Wilson[13] about the structure of the commutator near $x = y$, namely near the tip of the light cone. His conjecture is that there exists an expansion of the form

$$[J_\mu(x), J_\nu(y)] = \sum_{[\alpha]} C_{\mu\nu}{}^{[\alpha]}(x-y) F^{[\alpha]}(y), \tag{6}$$

where $C_{\mu\nu}{}^{[\alpha]}(x-y)$ are c-number functions, which include the singularities at $x = y$, and $F^{[\alpha]}(y)$ are operators. The index $[\alpha]$ includes internal group characterization and also Lorentz tensor structure. It is assumed that to any given degree of singularity near $x = y$, there are but a finite number of terms in the above expansion. The singularity of the function $C_{\mu\nu}{}^{[\alpha]}(x-y)$ is determined by the asymptotic "dimensionality" of the operators $F^{[\alpha]}(y)$ and the currents. For whenever there exists a scale operator $U(\lambda)$, such that

$$U(\lambda)A(x)U^{-1}(\lambda) = \lambda^{d(A)}A(\lambda x),$$

where $d(A)$ is a number associated with the operator A, then

$$C_{\mu\nu}{}^{[\alpha]}(\lambda x) = \lambda^{d(F^{[\alpha]}) - 2d(J)} C_{\mu\nu}{}^{[\alpha]}(x) \tag{7}$$

and $C_{\mu\nu}{}^{[\sigma]}$ is more regular near the origin the higher $d(F)$ $[d(J) = 3$, using the fact that the charges have dimensionality zero$]$. One should keep in mind that the numbers $d(A)$ are in general different than the dimension of the operators as appearing in a Lagrangian formulation.[13] Equation (7), together with the Lorentz covariance and locality of Eq. (6), fixes the C's up to constants. For example, for the part $C_{\mu\nu} = g_{\mu\nu}C$ and for $C(\lambda x) = \lambda^{2d}C(x)$ we get

$$C(x) \propto [(-x^2 + i\epsilon x_0)^2 - (-x^2 - i\epsilon x_0)^d]. \tag{8}$$

One may try and generalize the expansion (6) to the case of interest for us, namely near $(x-y)^2 = 0$, by postulating that for two currents J_a and J_b,

$$[J_a(x), J_b(y)] = \sum_{[\alpha]} C^{[\alpha]}(x-y) F^{[\alpha]}(x, y), \tag{9}$$

where a, b, and $[\alpha]$ include Lorentz and other quantum-number characterization, $F^{[\alpha]}(x,y)$ is regular at $(x-y)^2$, and $C^{[\alpha]}(x-y)$ includes the singularity near the light cone $(x-y)^2 = 0$.

We furthermore assume that $F^{[\alpha]}(x,y)$ is regular also at $x = y$,[14] so that we can determine the singularity of the C's from Wilson's expansion, since the $C^{[\alpha]}$ can be decomposed in terms of invariant functions, which depend on $(x-y)^2$ and $\epsilon(x_0 - y_0)$ only, and for those the singularity near $(x-y)^2 = 0$ or near $x = y$ is the same.

Let us assume, for simplicity, that the leading singularity on the light cone is of the form

$$[J(x), J(y)] = C(x-y)F(x,y) + \cdots, \tag{10}$$

where $F(x,y)$ is a scalar operator, $C(x)$ is as in (8), and we take, from now on, scalar currents for simplicity. Let us now take the matrix element between single-nucleon states, spin averaged, and look at the Bjorken scaling limit. We obtain

$$W(q^2, q\cdot p) = \int d^4x\, e^{iqx}\langle p|[J(x), J(0)]|p\rangle$$

$$\approx \int d^4x\, e^{iqx} C(x)\langle p|F(x, 0)|p\rangle.$$

Now, $\langle p|F(x,0)|p\rangle$ is a function of x^2 and $p\cdot x$. However, the most singular part of the matrix element of the commutator will be given if we take $\langle p|F(x,0)|p\rangle$ at $x^2 = 0$. We may therefore write

$$[\langle p|F(x,0)|p\rangle]_{x^2=0} = \int d\alpha\, g(\alpha) e^{i\alpha p\cdot x}, \tag{11}$$

and hence

$$W(q^2,\nu)\approx\int d\alpha\, g(\alpha)\int d^4x\, e^{i(q+\alpha p)\cdot x}C(x).$$

For $C(x)$ of the form of Eq. (8),

$$\int d^4x\, e^{ikx}C(x)\propto[(-k^2+i\epsilon k_0)^{-d-2}-(-k^2-i\epsilon k_0)^{-d-2}],$$

which shows that only $k^2\geq 0$ gives a nonvanishing integral. We get

$$W(q^2,\nu)\approx\int d\alpha\, g(\alpha)\{[-(q+\alpha p)^2+i\epsilon(q_0+\alpha p_0)]^{-d-2}-[-(q+\alpha p)^2-i\epsilon(q_0+\alpha p_0)]^{-d-2}\}.$$

From the fact that $W(q^2,\nu)$ vanishes whenever $q^2+2M\nu<0$ and $q^2-2M\nu<0$, we get that only $-1\leq\alpha\leq 1$ can appear. We also have $g(\alpha)=g(-\alpha)$, since W is odd in ν. Therefore, for $\nu>0$,

$$W(q^2,\nu)\approx(2M\nu)^{-d-2}\int_{-1}^{1}d\alpha\, g(\alpha)[(\omega-\alpha+i\epsilon)^{-d-2}-(\omega-\alpha+i\epsilon)^{-d-2}],\qquad(12)$$

with $\omega=-q^2/2M\nu$ (for inelastic ep scattering we have of course $\nu=0$). This would be our generalized scaling law: $W(q^2,\nu)\rightarrow\nu^{-d-2}f(\omega)$ in the limit $\nu\rightarrow\infty$ and $\omega=-q^2/2M\nu$ fixed. If Eq. (12) holds also in the deep Regge limit, in which case $\omega\rightarrow 0$, we need, to ensure that $W(q^2,\nu)\rightarrow\beta(q^2)\nu^{\alpha(0)}$, that $f(\omega)\sim\omega^{-[d+2+\alpha(0)]}$ for $\omega\rightarrow 0$. $\alpha(0)$ is the intercept at $t=0$ of the leading Regge trajectory [for the case of vector currents we would have $\alpha(0)-2$ instead of $\alpha(0)$ for W_2]. This is achieved by a $g(\lambda)$ which behaves like $|\lambda|^{-[\alpha(0)+1]}$ for small λ [it is easy to see that by changing $\alpha\rightarrow\alpha\omega$ in Eq. (12)]. Comparing with Eq. (11) we see that Regge behavior is connected to the large $p\cdot x$ behavior of our matrix element. The singularity on the light cone thus determines the generalized scaling law, which when combined with the large $p\cdot x$ behavior gives the Regge behavior. For the case of vector currents, Bjorken's scaling laws[1] are that W_1 and νW_2 are functions of ω only. The Regge behavior is $\nu^{\alpha(0)}$ for W_1 and $\nu^{\alpha(0)-2}$ for W_2, with the leading singularity being the Pomeranchukon.[2,3] When applying the expansion (9) to vector currents,[15] we therefore need a term whose contribution to W_2 would be given by a singularity $\epsilon(x_0)\delta(x^2)$ for $C(x)$ and a $|\lambda|^{1-\alpha(0)}$ behavior for $g(\lambda)$ near $\lambda\approx 0$, and a term for W_1 with $C(x)$ of the form $\epsilon(x_0)\delta'(x^2)$ and $g(\lambda)$ behaving like $|\lambda|^{-\alpha(0)-1}$.

Going back to $W(q^2,\nu)$ of Eq. (12), we see that in the scaling limit we have $\nu^{-d-2}f(\omega)$ and in the deep Regge limit $\nu^{\alpha(0)}(-q^2)^{-[d+2+\alpha(0)]}$. Following arguments based on duality,[16] we would expect the decrease in q^2 to be more rapid for the non-Pomeranchuk contribution. This would mean an increase in d for a non-Pomeranchuk term which is more than the decrease in $\alpha(0)$. This in turn means that the lower the intercept of the Regge trajectory, the less singular the behavior near the light cone. For the difference between electron-proton and electron-neutron inelastic scattering we thus expect W_1 and W_2 to behave like $\nu^{-d_1-1}f_1(\omega)$ and $\nu^{-d_2-2}f_2(\omega)$, respectively, with $d_n+\alpha_{A_2}(0)>0$, $n=1,2$, where $\alpha_{A_2}(0)$ is the zero intercept of the A_2 trajectory which is supposed to be the leading Regge contribution for the above mentioned difference.[17]

Note that for a certain term in the expansion Eq. (9), the t dependence for a nonforward matrix element is given only through the matrix element $\langle p|F^{[\alpha]}(x,0)|p'\rangle$ of the regular operator $F^{[\alpha]}(x,0)$. There is certainly no t dependence to the part singular on the light cone. We could of course put some t dependence to the singular part by assuming

$$\langle p|[J_a(x),J_b(0)]|p'\rangle=\sum_{[\alpha]}C^{[\alpha]}(x,t)F^{[\alpha]}(p\cdot x,p'\cdot x,t),\qquad(13)$$

with $F^{[\alpha]}$ regular at $x=0$ and $C^{[\alpha]}$ of the form

$$C(x,t)\sim[(-x^2+i\epsilon x_0)^{d(t)}-(-x^2-i\epsilon x_0)^{d(t)}].\qquad(14)$$

However, Eq. (13) contradicts the Wilson expansion Eq. (6) when we take $x\approx 0$ in the former and assume a nonconstant $d(t)$.

Let us now comment on e^+e^- annihilation into hadrons. We know, from the discussion regarding W_1 in inelastic ep scattering, that the leading singularity near the light cone is a $\delta'(x^2)$ singularity. Therefore we would expect $\langle 0|[J_\mu(x),J_\nu(0)]|0\rangle$ to be more singular than that.[18] Defining

$$\langle 0|[J_\mu(x),J_\nu(0)]|0\rangle=\int d^4q\, e^{-iqx}\epsilon(q_0)\rho(q^2)(-g_{\mu\nu}q^2+q_\mu q_\nu)\qquad(15)$$

we would get that $\rho(s)$ decreases slower than s^{-1} as $s\rightarrow\infty$. The Schwinger term, being proportional to $\int\rho(s)ds$, would therefore diverge. The annihilation cross section $\sigma_{e^+e^-}(s)$ of e^+e^- into hadrons, which is proportional to $\rho(s)/s$, would decrease slower than s^{-2}.

Finally let us remark that expansions of the form of Eq. (9) can be extended to the cases of time-ordered products or ordinary products of operators, changing appropriately the structure of the singular functions on the light cone. It should be emphasized that our expansion, Eq. (9), as well as Wilson's, are weak equalities, namely holding for matrix elements between fixed states.

In a case like proton + proton → $(\mu^+\mu^-)$ + hadrons, in the limit $s\to\infty$, Q^2/s fixed (Q^2 is the mass of the $\mu^+\mu^-$ pair), the approach to the light cone involves a simultaneous change in the states and thus a more careful investigation is required.[19]

It is a pleasure to thank Professor D. Zwanziger for many fruitful and illuminating discussions during the course of this work. Discussions with Professor Drell, Professor Harari, and Professor Kugler are also gratefully acknowledged.

---

*Research supported in part by the U. S. Air Force Office of Scientific Research through the European Office of Aerospace Research under Contract No. F-61052-68-C-0070.

[1]J. D. Bjorken, Phys. Rev. 179, 1547 (1969).

[2]H. D. I. Abarbanel, M. L. Goldberger, and S. B. Treiman, Phys. Rev. Lett. 22, 500 (1969).

[3]H. Harari, Phys. Rev. Lett. 22, 1078 (1969), and 24, 286 (1970).

[4]J. D. Bjorken and E. A. Paschos, Phys. Rev. 185, 1975 (1969).

[5]R. P. Feynman, Phys. Rev. Lett. 23, 1415 (1969).

[6]S. D. Drell, D. J. Levy, and T. M. Yan, Phys. Rev. Lett. 22, 744 (1969), and Phys. Rev. 187, 2159 (1969), and Phys. Rev. D 1, 1035, 1617 (1970); S. D. Drell and T. M. Yan, Phys. Rev. Lett. 24, 181 (1970), and Stanford Linear Accelerator Center Report No. SLAC-PUB-692 (unpublished).

[7]R. A. Brandt, Phys. Rev. Lett. 22, 1149 (1969), and *ibid.* 23, 1260 (1969).

[8]G. Altarelli and H. R. Rubinstein, Phys. Rev. 187, 2111 (1969).

[9]R. Jackiw, R. Van Royen, and G. B. West, Phys. Rev. D (to be published).

[10]L. S. Brown, "Causality in Electroproduction at High Energy" (to be published).

[11]F. J. Gilman, in *Proceedings of the International Symposium on Electron and Photon Interactions at High Energies, Liverpool, England, September 1969,* edited by D. W. Braben (Daresbury Nuclear Physics Laboratory, Daresbury, Lancashire, England, 1970).

[12]For a review of the experimental data see R. Taylor, in *Proceedings of the International Symposium on Electron and Photon Interactions at High Energies, Liverpool, England, September 1969,* edited by D. W. Braben (Daresbury Nuclear Physics Laboratory, Daresbury, Lancashire, England, 1970).

[13]K. G. Wilson, Phys. Rev. 179, 1499 (1969).

[14]An expansion of the form Eq. (9) certainly holds in free-field theories. For example, taking $j(x)=\varphi^\dagger(x)\varphi(x)$, where $\varphi(x)$ is a scalar field, we get $[j(x),j(y)]=i\Delta(x-y)\Delta_1(x-y)+i\Delta(x-y)[:\varphi^\dagger(x)\varphi(y):+:\varphi^\dagger(y)\varphi(x):]$, where $\Delta(x)=-[i/(2\pi)^3]\int d^4p\, e^{-ipx}\epsilon(p_0)\delta(p^2-m^2)$, $\Delta_1(x)=(2\pi)^{-3}\times\int d^4p\, e^{-ipx}\delta(p^2-m^2)$, and colons denote normal ordering. {The leading singularity of the c number on the right-hand side is $-(i/8\pi^3)\epsilon(x_0-y_0)\delta'[(x-y)^2]$ and that of $\Delta(x)$ is $-(2\pi)^{-1}\epsilon(x_0)\delta(x^2)$.} It is interesting to note that when calculating $[J_\mu(x),J_\nu(y)]$ for the free Dirac case, and then going over to equal-time commutators by the method of Wilson (Ref. 13), one gets a Schwinger term diverging like $(x_0-y_0)^{-2}$ even though no care was exercised in defining the currents by point separation.

[15]Note that when using an expansion of the form Eq. (9) for vector currents, we may arrange that W_1 and νW_2 have different scaling properties by giving different singularities to the various terms in the expansion. The possibility of different scaling properties for W_1 and νW_2 was considered in the literature. See, for example, Ref. (8).

[16]See the first paper in Ref. (3).

[17]Such a behavior would give a convergent result for the dispersive contribution Δm^I of the n-p mass difference. See H. Harari, Phys. Rev. Lett. 17, 1303 (1966). The contribution of the subtraction term $\Delta M^{\rm sub}$ remains to be determined.

[18]Arguments to this effect will be given elsewhere. In case there exists an asymptotic "dimensionality," it has to be $d=3$ for the currents, and then $\rho(s)$ would be constant at high s.

[19]After this note was typed we were informed by Professor Drell that our generalization of Wilson's expansion to the light cone has also just been suggested by G. Altarelli, R. A. Brandt, and G. Preparata.