ULTIMATE TEMPERATURE AND THE EARLY UNIVERSE*

Kerson Huang and Steven Weinberg
Laboratory for Nuclear Science and Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 5 August 1970)

The early history of the universe is discussed in the context of an exponentially rising density of particle states.

There are now plausibe theoretical models[1] for the thermal history of the universe back to the time of helium synthesis, when the temperature was 0.1 to 1 MeV. Our present theoretical apparatus is really inadequate to deal with much earlier times, say when $T \gtrsim 100$ MeV, and in lieu of any better ideas it is usual to treat the matter of the very early universe as consisting of a number of species of essentially free particles. But how many species?

At one extreme, it might be assumed that the number of particle species stays fixed (perhaps just quarks, antiquarks, leptons, antileptons, photons, and gravitons). In this case, the temperature T will vary with the cosmic scale factor[2] $R(t)$ according to the relation $T \propto 1/R$. The present universe should then contain various relics of the early inferno: There sould be a 1°K blackbody gravitational radiation,[3] if TR stayed roughly constant between the times that the gravitons and the photons decoupled from the rest of the universe; also, according to Zeldovich,[4] the leftover quarks should be about as common as gold atoms. The gravitational radiation would not have been seen, but the quarks would have been, unless, of course, quarks do not exist.

At the other extreme, one might assume that the number of species of particles with mass between m and $m+dm$ increases as $m \to \infty$ as fast as possible:

$$N(m)dm \to Am^{-B}e^{\beta_0 m}dm. \tag{1}$$

If $N(m)$ increased any faster, the partition function would not converge. With the increase (1), the partition function converges only if the temperature[5] is less than $1/\beta_0$. The quantity $T_0 \equiv 1/\beta_0$ is thus a maximum temperature for any system in thermal equilibrium.

Support for this latter sort of model comes from two quite different directions:

(1) The transverse momentum distribution of secondaries in very high energy collisions is observed to be roughly $\exp(-|p_\perp|/160 \text{ MeV})$. Hagedorn[6] interprets this distribution in terms of a statistical model with $T_0 \simeq 160$ MeV and $B = \frac{5}{2}$.

(2) If particles fall on families of parallel linearly rising Regge trajectories, their masses take discrete values $m_1, m_2, \cdots$, where

$$\alpha' m_n^2 + \alpha_0 = n. \tag{2}$$

Here $\alpha' \approx 1 \text{ GeV}^{-2}$ is the universal Regge slope and α_0 is a number, of order unity, characterizing the family. The extension of the Veneziano model[7] to multiparticle reactions requires[8] that the number of particle states with mass m_n equals the degeneracy of the eigenvalue n of the operator

$$N = \sum_{\mu=1}^{D} \sum_{k=1}^{\infty} k a_{\mu k}^\dagger a_{\mu k}, \tag{3}$$

where $a_{\mu k}$ and $a_{\mu k}^\dagger$ are an infinite set of annihilation and creation operators. For $n \to \infty$, this number is[9]

$$P_{nD} \to 2^{-1/2}(D/24)^{(D+1)/4} n^{-(D+3)/4} \times \exp\{2\pi(\tfrac{1}{6}Dn)^{1/2}\}. \tag{4}$$

Equations (2) and (4) lead to an asymptotic level density of form (1), with

$$\beta_0 = 2\pi(\tfrac{1}{6}D\alpha')^{1/2}, \quad B = \tfrac{1}{2}(D+1). \tag{5}$$

The value of D is not certain—originally Fubini and Veneziano[8] had $D=4$, but Lovelace[10] argues that D is larger, possibly $D=5$.

Table I summarizes the values of T_0 and B for these various models. Lovelace[10] has emphasized the striking agreement between the values of T_0 derived in such different ways. We now see that

Table I. Possible values of the parameters in the level-density formula (1).

Model	$T_0 \equiv 1/\beta_0$	B
(1) Hagedorn[a]	~160 MeV	$\frac{5}{2}$
(2) Veneziano[b] (with $\alpha' = 1 \text{ GeV}^{-2}$)		
$D=4$	180 MeV	$\frac{5}{2}$
$D=5$	174 MeV	3
$D=6$	159 MeV	$\frac{7}{2}$
$D=7$	147 MeV	4

[a]Ref. 6. [b]Ref. 8.

even the values of B are the same, or nearly so. In addition, the Veneziano model approximates the hadronic S matrix as a sum over narrow resonances, and since all these resonances are included in (1), it is perhaps not unreasonable to ignore their interactions. All of this leads us to suspect that a free-particle model with level density (1) may not be a bad approximation at high temperature.

Let us first apply this model to the relatively simple case of a universe with zero net baryon number.[11] In an adiabatic expansion, the only conserved quantity is the entropy per unit coordinate volume,[12]

$$s=(\rho+p)R^3/T, \qquad (6)$$

where p and ρ are the pressure and the density of energy per unit proper volume. By integrating the usual Fermi and Bose distribution functions over m with weighting function (1), we find, as $T\to T_0$,

$$\rho\propto\begin{cases}(T-T_0)^{B-7/2} & \text{for } B<\tfrac{7}{2},\\ |\ln(T-T_0)| & \text{for } B=\tfrac{7}{2}.\end{cases} \qquad (7)$$

For $B>\frac{7}{2}$, ρ is finite even at $\beta=\beta_0$. The asymptotic behavior of the pressure is the same, except that B is replaced with $B+1$, so that $p\ll\rho$ for $B\leq\frac{7}{2}$. Hence, for $B\leq\frac{7}{2}$, Eq. (6) requires that T should approach T_0 according to

$$T_0-T\propto\begin{cases}R^{6/(7-2B)} & \text{for } B<\tfrac{7}{2},\\ \exp(\text{const}/R^3) & \text{for } B=\tfrac{7}{2}.\end{cases} \qquad (8)$$

However, for $B>\frac{7}{2}$ there is no way that (6) can remain constant in the limit $R\to 0$. If we take our model seriously, then for zero baryon number and $B>\frac{7}{2}$, thermal equilibrium becomes impossible when $R(t)$ is less than some small but finite value.

Now let us consider the effect of a nonzero net baryon-number density n_B. We now have two conservation laws, for the baryon number per unit coordinate volume[12]

$$\nu=n_BR^3, \qquad (9)$$

and for the entropy per baryon

$$\sigma\equiv s/\nu=[p+\rho-\mu n_B]/n_BT, \qquad (10)$$

where μ is the baryonic chemical potential. In order to keep (9) constant as $R\to 0$, we must have $\mu\to\infty$ or $T\to T_0$ or both. There is no way that (10) can be kept constant if $T\to T_0$, either with μ constant or $\mu(T-T_0)$ constant. However, a solution does exist in which T approaches a finite value less than T_0 as $\mu\to\infty$. The integrals for ρ, p, and n_B are dominated in this limit by the contribution from within a finite distance of the Fermi surface at energy μ, and we find

$$\rho\to A'e^{\mu/T_0}\mu^{5/2-B}T\csc(\pi T/T_0)\{1+(\pi T/\mu)(B-\tfrac{5}{2})\cot(\pi T/T_0)+(3T_0/2\mu)(B-\tfrac{1}{4})+O(\mu^{-2})\}, \qquad (11)$$

$$p\to A'e^{\mu/T_0}\mu^{5/2-B}T\csc(\pi T/T_0)\{T_0/\mu+O(\mu^{-2})\}, \qquad (12)$$

$$n_B\to A'e^{\mu/T_0}\mu^{3/2-B}T\csc(\pi T/T_0)\{1+(\pi T/\mu)(B-\tfrac{3}{2})\cot(\pi T/T_0)+(3T_0/2\mu)(B-\tfrac{1}{4})+O(\mu^{-2})\}, \qquad (13)$$

where $A'=(T_0^{\,3}/8\pi)^{1/2}A$. The specific entropy (10) then approaches a limit,

$$\sigma\to(T_0/T)-\pi\cot(\pi T/T_0)+O(\mu^{-1}). \qquad (14)$$

The observed values of the present blackbody radiation temperature and baryon number density indicate[13] that σ is large, roughly of order 10^8. Equation (14) then requires that as $R\to 0$, the temperature approaches a value T with

$$(T_0-T)/T_0\simeq 1/\sigma\approx 10^{-8}. \qquad (15)$$

It is somewhat of a mystery why σ is so large. Unfortunately, Eq. (14) only replaces this mystery with another one: Why does the universe begin with a temperature so very close to T_0?

The baryon conservation law (9), together with (13), gives the growth of μ as $R\to 0$,

$$\mu\to 3T_0|\ln R|+O(\ln|\ln R|). \qquad (16)$$

The mean square particle velocity then vanishes for $R\to 0$ like

$$\langle v^2\rangle\to 3p/\rho\to 3T_0/\mu\to 1/|\ln R|. \qquad (17)$$

Since the energy density is dominated by nonrelativistic particles, we expect stability against local gravitational collapse, as for a polytrope with $\gamma=\frac{5}{3}$. A peculiar feature of these results is that the mean particle mass becomes much larger than the temperature,

$$\langle m\rangle\to\rho/n_B\to\mu\to 3T_0|\ln R|. \qquad (18)$$

The behavior of $R(t)$ as $t\to 0$ (defined as the time when $R=0$) is given here by Einstein's equations as

$$R\propto t^{2/3}|\ln t|^{1/3},$$

instead of the behavior $R\propto t^{2/3}$ or $R\propto t^{1/2}$ expected in a Friedmann or a Tolman model.

A curious tentative view of cosmic history emerges from these considerations: (1) From the present back to when $T \approx 1$ eV, the energy density ρ has been dominated by nonrelativistic baryons. (2) Between $T \approx 1$ MeV and $T \approx 1$ eV, ρ was dominated by photons. (3) Between $T \approx 0.9T_0$ and $T \approx 1$ MeV, ρ was dominated by leptons, antileptons, and photons. (4) Between $T_0 - T \approx 10^{-8}T_0$ and $T \approx 0.9T_0$, ρ was dominated by nonrelativistic mesons, baryons, and antibaryons, the latter two in nearly equal numbers. (5) At earlier times, ρ was dominated, once again, by nonrelativistic baryons! The transition between eras (3) and (4), and eras (4) and (5), will be studied numerically in a future publication.

---

*Work supported in part through funds provided by the Atomic Energy Commission under Contract No. AT(30-1)-2098.

[1]For reviews, see Ya. B. Zeldovich, Usp. Fiz. Nauk 89, 647 (1966) [Sov. Phys. Usp. 9, 602 (1967)]; I. D. Novikov and Ya. B. Zeldovich, Ann. Rev. Astron. Astrophys. 5, 627 (1967); E. R. Harrison, Phys. Today 21, No. 6, 31 (1968).

[2]The quantity $R(t)$ is the scale factor appearing in the usual Robertson-Walker metric. See, e.g., H. Bondi, *Cosmology* (Cambridge Univ., Cambridge, England, 1960), p. 102.

[3]J. A. Wheeler, in *La Structure et l'Evolution de l'Univers* (Institut International de Physique Solvay, Bruxelles, Belgium, 1958), p. 96; Ya. B. Zeldovich, in *Advances in Astronomy and Astrophysics* (Academic, New York, 1965), p. 319; F. Winterberg, Nuovo Cimento 53B, 264 (1968); S. Weinberg, Contemp. Phys. 1, 559 (1969); R. A. Matzner, Astrophys. J. 154, 1123 (1968); etc.

[4]Ya. B. Zeldovich, Comments Astrophys. Space Phys. 11, 12 (1970).

[5]We use units with the Boltzmann constant equal to unity, so all temperatures have the dimensions of energy, and the entropy is dimensionless.

[6]R. Hagedorn, Nuovo Cimento Suppl. 3, 147 (1965), and 6, 311 (1968), and Nuovo Cimento 52A, 1336 (1967), and 56A, 1027 (1968); R. Hagedorn and J. Ranft, Nuovo Cimento Suppl. 6, 169 (1968). Limiting temperatures have also been considered by Yu. B. Rumer, Zh. Eksp. Teor. Fiz. 38, 1899 (1960) [Sov. Phys. JETP 11, 1365 (1960)].

[7]G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[8]S. Fubini and G. Veneziano, Nuovo Cimento 64A, 811 (1969); K. Bardakçi and S. Mandelstam, Phys. Rev. 184, 1640 (1969); S. Fubini, D. Gordon, and G. Veneziano, Phys. Lett. 29B, 679 (1969). This work actually refers explicitly to meson families; we are assuming here that the baryon level density has the same behavior for $n \to \infty$.

[9]This result is a generalization of the asymptotic formula for the "partitio numerorum" P_{n1} derived by G. H. Hardy and S. Ramanujan, Proc. London Math. Soc. 17, 75 (1918). Our derivation is based on an examination of the properties of the generating function $G_D(z)$, defined as $\sum_n P_{nD} z^n$. The asymptotic behavior of $G_D(z)$ as $z \to 1^-$ can immediately be obtained from the asymptotic behavior of $G_1(z)$ given by Hardy and Ramanujan, because $G_D(z) = [G_1(z)]^D$. The asymptotic behavior of P_{nD} as $n \to \infty$ can then be determined by using a saddle-point approximation to evaluate the contour integral of $z^{-n-1}G_D(z)$ around the origin. The result so obtained reduces to that of Hardy and Ramanujan for $D = 1$.

[10]C. Lovelace, in Proceedings of the Conference on Regge Poles, Irvine, Calif., 6 December 1969 (unpublished).

[11]This case has already been considered by R. Hagedorn, Astron. Astrophys. 5, 184 (1970), for $B = \frac{5}{2}$. Our results agree with his for this special case. However, it turns out to make an extraordinary difference if there is any net baryon-number density, or, for zero baryon number, if $B \geq \frac{7}{2}$.

[12]By "coordinate volume" we mean volume in the comoving Robertson-Walker coordinate system; see Bondi, Ref. 2. The ratio of proper to coordinate volume is $R^3(t)$.

[13]The value of σ is estimated from the present ratio of the number densities of blackbody photons and nucleons; see Ref. 1. In using this estimate, we are tacitly neglecting the production of entropy by neutrino viscosity and other dissipative effects; see C. W. Misner, Nature 214, 40 (1967), and Astrophys. J. 151, 431 (1968), and Phys. Rev. Lett. 19, 533 (1967); E. L. Schücking and E. A. Spiegel, Comments Astrophys. Space Phys. 2, 121 (1970).