ASYMPTOTIC SYMMETRY, PARTICLE MIXING, AND K_{l3}-DECAY BRANCHING RATIOS

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A concept of asymptotic symmetry is formulated and applied to the determination of mixing parameters between the π^0, η^0, and $\eta'^0(958)$ in broken SU(3) and SU(2) symmetries. One of the solutions gives rise to a rather large violation of the $|\Delta \vec{I}|=\frac{1}{2}$ rule in the K_{l3} decays which is not inconsistent with present experiment.

This paper aims to discuss the particle mixing effects by taking the view of asymptotic symmetry which assumes that the SU(3) and SU(2) symmetries are well realized among particles of extremely high momenta. A similar assumption has been used in deriving several successful sum rules from the chiral SU(3)⊗SU(3) algebra.[1,2] To derive our result by a short-cut (but instructive) computation, we express the requirement of asymptotic SU(3) symmetry in a simple form. Let us consider the pseudoscalar nonet and denote their annihilation operators as $a_\alpha(\vec{k})$, where α stands for $\pi^{\pm,0}$, $K^{+,0}$, $\bar{K}^{+,0}$, η^0, and $\eta'^0(958)$, and $\vec{k}$ denotes their momenta. It should be noted that $a_\alpha(\vec{k})$ are not the Heisenberg operators but the operators of the physical (i.e., incoming) particles with mass m_α. Denoting the SU(3) generators by V_i, the transformation of physical particles in broken SU(3) symmetry can be expressed in the following form:

$$[V_i, a_\alpha(\vec{k})] = i\textstyle\sum_\beta u_{i\alpha\beta}(\vec{k})a_\beta(\vec{k}) + \delta u_{i\alpha}(\vec{k}). \qquad (1)$$

Here, the first term on the right-hand side of Eq. (1) picks up all the terms linear in a's (but not $a^\dagger$'s) and the remainder is denoted by δu. The δu is of the order of SU(3) breaking. By inspecting the vacuum expectation value of the Jacobi identity for $[a_\beta^\dagger(\vec{k}),\ V_i, a_\alpha(\vec{k})]$, we find a constraint,

$$u_{i\alpha\beta}^*(\vec{k}) = -u_{i\beta\alpha}(\vec{k}). \qquad (2)$$

Our requirement of asymptotic symmetry is that the $\delta u_{i\alpha}(\vec{k})$ can always be neglected[3] in the limit $|\vec{k}|\to\infty$. Let us now recall that the total Hamiltonian, when expressed in terms of the physical fields, takes the form

$$H = \textstyle\sum_\alpha \int \omega_\alpha(\vec{k})a_\alpha^\dagger(\vec{k})a_\alpha(\vec{k})d^3k + \cdots, \qquad (3)$$

with $\omega_\alpha(\vec{k}) = (\vec{k}^2 + m_\alpha^2)^{1/2}$. For large $|\vec{k}|$ we can write

$$\omega_\alpha(\vec{k})a_\alpha^\dagger(\vec{k})a_\alpha(\vec{k}) = |\vec{k}|a_\alpha^\dagger(\vec{k})a_\alpha(\vec{k}) + (m_\alpha^2/2|\vec{k}|)a_\alpha^\dagger(\vec{k})a_\alpha(\vec{k}) + \cdots.$$

By utilizing Eq. (2) and ignoring the δu term in Eq. (1) from our asymptotic condition, we find $[V_i, |\vec{k}|\sum_\alpha a_\alpha^\dagger(\vec{k})a_\alpha(\vec{k})] = 0$ for $|\vec{k}|\to\infty$. We can thus proceed for large $|\vec{k}|$ as follows:

$$\begin{aligned} i\langle\vec{k},\alpha|[V_i,\dot{V}_j]|\vec{k},\alpha\rangle &= \langle\vec{k},\alpha|[V_i,[V_j,H]]|\vec{k},\alpha\rangle \\ &= \langle\vec{k},\alpha|[V_i,[V_j,\textstyle\sum_\alpha(m_\alpha^2/2|\vec{k}|)a_\alpha^\dagger(\vec{k})a_\alpha(\vec{k})]]|\vec{k},\alpha\rangle. \end{aligned} \qquad (4)$$

Now Eq. (1) shows that the operators $a_\alpha(\vec{k})$ form a linear realization of the SU(3) group for large $|\vec{k}|$. Usually the linear SU(3) representation is expressed in terms of $a_j(\vec{k})$ $[j=1,2,\cdots,9]$ which satisfy

$$[V_i, a_j(\vec{k})] = if_{ijl}a_l(\vec{k}) \text{ for } j=1,\cdots,8,$$
$$= 0 \text{ for } j=9. \qquad (5)$$

Then our $a_\alpha(\vec{k})$ must be linearly related to the $a_j(\vec{k})$ in the limit $|\vec{k}|\to\infty$,

$$a_\beta(\vec{k}) = \textstyle\sum_j c_{\beta j}a_j(\vec{k}), \quad |\vec{k}|\to\infty. \qquad (6)$$

$c_{\beta j}$ defines the mixing angles. In our formulation the mixing angles are always defined in the asymptotic limit.[4] We note that at finite momentum the situation is far more involved since the mixing parameters will be influenced by the δu term in Eq. (1). In broken SU(3) and SU(2) symmetry the mixing appears among the operators of the π^0, η^0, and η'^0 fields. By using the Euler angles $(\omega, \theta, \varphi)$ we write for $|\vec{k}|\to\infty$, for example, $a_{\pi^0}(\vec{k}) = \cos\theta\cos\varphi a_3(\vec{k}) + (\sin\theta\cos\omega - \cos\theta\sin\varphi\sin\omega)a_8(\vec{k}) + (\sin\theta\sin\omega + \cos\theta\sin\varphi\cos\omega)a_9(\vec{k})$, etc. Thus the $c_{\beta j}$ can be expressed in terms of the following mixing parameters: $\alpha \equiv c_{\pi 3} = \cos\theta \cos\varphi$, $\beta \equiv c_{\pi 8} = \sin\theta \cos\omega - \cos\theta \sin\varphi \sin\omega$, $\gamma \equiv c_{\pi 9} = \sin\theta \sin\omega + \cos\theta \times \sin\varphi \cos\omega$, $\beta' \equiv -c_{\eta 3} = \sin\theta \cos\varphi$, $\gamma' \equiv -c_{\eta' 3} = \sin\varphi$, $a \equiv c_{\eta 8} = \cos\theta \cos\omega + \sin\theta \sin\varphi \sin\omega$, $b \equiv c_{\eta 9} = \cos\theta \sin\omega - \sin\theta \sin\varphi \cos\omega$, $c \equiv c_{\eta' 8} = -\cos\theta \sin\omega$, and $d \equiv c_{\eta' 9} = \cos\varphi \cos\omega$. All other $c_{\beta j}$ are zero. If we assume, as usual, that both the SU(3)- and SU(2)-breaking interactions belong to an SU(3) octet, the following "exotic" commutation relations hold[2] for the V's: $[V_{\pi^+}, \dot{V}_{\pi^+}] = 0$, $[V_{K^0}, \dot{V}_{\pi^-}] = 0$, and $[V_{K^0}, \dot{V}_{K^0}] = 0$. Then using Eqs. (4) and (5), these commutators lead to the following mass relations:

$$(1-\beta^2-\gamma^2)(m_{\pi^0}{}^2 - m_{\pi^+}{}^2) + (\beta')^2(m_\eta{}^2 - m_{\pi^+}{}^2) + (\gamma')^2(m_{\eta'}{}^2 - m_{\pi^+}{}^2) = 0, \tag{7}$$

$$(m_{K^0}{}^2 - m_{K^+}{}^2) + 3^{1/2}(1-\beta^2-\gamma^2)^{1/2}\beta(m_{\pi^0}{}^2 - m_{\pi^+}{}^2) - 3^{1/2}a\beta'(m_\eta{}^2 - m_{\pi^+}{}^2) - 3^{1/2}c\gamma'(m_{\eta'}{}^2 - m_{\pi^+}{}^2) = 0, \tag{8}$$

$$(\beta' + 3^{1/2}a)^2(m_{K^0}{}^2 - m_\eta{}^2) + (\gamma' + 3^{1/2}c)^2(m_{K^0}{}^2 - m_{\eta'}{}^2) + [(1-\beta^2-\gamma^2)^{1/2} - 3^{1/2}\beta]^2(m_{K^0}{}^2 - m_{\pi^0}{}^2) = 0. \tag{9}$$

Here β, γ, β', and γ' are the quantities of the order of SU(2) breaking. In the SU(2) limit ($\beta' = \gamma' = \beta = \gamma = 0$, $\theta = \varphi = 0$), $m_{\pi^0} = m_{\pi^+}$ and $m_{K^0} = m_{K^+}$ from Eqs. (7) and (8), respectively, while Eq. (9) gives the Gell-Mann–Okubo (GMO) mass formula, including the SU(3) η-η' mixing (mixing angle ω), i.e., $4m_K{}^2 - 3m_\eta{}^2 - m_\pi{}^2 - 3\sin^2\omega(m_{\eta'}{}^2 - m_\eta{}^2) = 0$. In fact, Eq. (9) provides the modified GMO mass formula, including SU(2) breaking.[2] In the presence of SU(2) breaking, Eqs. (7) and (8) become relevant. If we use the same approach[2] for the hyperons, the sum rules corresponding to Eqs. (7) and (8) are the well-known ones,[5] $m_{\Sigma^-} - m_{\Sigma^0} = m_{\Sigma^0} - m_{\Sigma^+}$ and $(m_n - m_p) + (m_{\Xi^-} - m_{\Xi^0}) = (m_{\Sigma^-} - m_{\Sigma^+})$, respectively. The main difference, besides the fact that we have a nonet for the bosons, is that the Σ^0-Λ^0 mixing is not very important.[2] The origin of these three sum rules is clear. They are the conditions (evaluated in the asymptotic limit) that the $I = 2$, 1, and 0 parts of the $\underline{27}$ representation do not appear in the total Hamiltonian, Eq. (3), leaving us only the octet term. In the absence of the η' ($\theta = \omega = 0$), Eq. (8) gives the η-π transition mass previously obtained.[6] Equations (7), (8), and (9) are just sufficient to determine all the mixing parameters. However, (without using a computer) we here solved the equations in an approximate way by assuming that the SU(3) η-η' mixing is larger than the SU(2) mixing. Namely, we take $a \simeq \cos\omega$, $b \simeq \sin\omega$, $c = -\sin\omega$, $d = \cos\omega$, $\beta' \simeq \beta\cos\omega + \gamma\sin\omega$, and $\gamma' \simeq -\beta\sin\omega + \gamma\cos\omega$. From the SU(3) GMO mass formula, the SU(3) η-η' mixing angle is given by $\sin\omega \simeq \pm 0.18$. This rather small value of ω makes our approximation crude. We obtain the following sets of values of β and γ: (I) $\sin\omega = 0.18$, $\beta = 0.022$, $\gamma = 0.038$; (II) $\sin\omega = 0.18$, $\beta = -0.0064$, $\gamma = -0.038$; (III) $\sin\omega = -0.18$, $\beta = 0.019$, $\gamma = -0.032$; and (IV) $\sin\omega = -0.18$, $\beta = -0.0061$, $\gamma = 0.038$. Because of our approximation, only the first figures of the numbers for β and γ may be trusted. For β we thus essentially have two distinct values, one about 0.02 and the other about -0.006. The latter is close in magnitude to the one obtained[6] without considering the η', while the former is considerably larger and may have new interesting implications. We now point out that the larger value of β obtained above has an appreciable effect on the violation of the $|\Delta\vec{I}| = \frac{1}{2}$ rule in the K_{l3} decays. We consider here only the K_{e3} decays since for these decays we can neglect the contribution of the so-called f_- form factors. The relevant matrix elements of the $K_{e3}{}^+$ and $K_{Le3}{}^0$ decays are then

$$\langle \pi^0(\vec{p}\,')|V_\mu{}^{K^-}(0)|K^+(\vec{p})\rangle = (2p_0 2p_0{}')^{1/2}(-1)2^{1/2}(p+p')_\mu f_+(q^2)$$

and

$$\langle \pi^-(\vec{p}\,')|V_\mu{}^{K^-}(0)|K^0(\vec{p})\rangle = (2p_0 2p_0{}')^{1/2}(-1)(p+p')_\mu g_+(q^2).$$

Here $q^2 = (p-p')^2$. In the absence of SU(2) breaking, $f_+(q^2) = g_+(q^2)$ and, in particular, $f_+(0) = g_+(0) = 1 + O(\epsilon^2) + \cdots$. Here $O(\epsilon^2)$ denotes the second-order SU(3)-breaking effect. The $O(\epsilon^2)$ term will be small [in our asymptotic SU(3) symmetry it is taken to be zero], and there is a plausible argument[7] that $O(\epsilon^2) < 0$. The pion energy spectra of the K_{e3} decays have a maximum around $q^2 = 0$. Therefore, the most important contribution to the K_{e3}-decay rates comes from the form factors in the region around $q^2 = 0$. On the other hand, our asymptotic symmetry (which is realized in the limit $|\vec{p}| = |\vec{p}\,'| = \infty$) yields best information for the form factors with $q^2 \simeq 0$. We now note that in the presence of SU(2) breaking the $f_+(q^2)$ will be renormalized by mixing. Since we can write for large $|\vec{p}\,'|$, approximately, $|\pi^0(\vec{p}\,')\rangle \simeq a_3{}^\dagger(\vec{p}\,')|0\rangle + \beta a_3{}^\dagger(\vec{p}\,')|0\rangle + \gamma a_9{}^\dagger(\vec{p}\,')|0\rangle$, we obtain using Eq. (5) $f_+(0) \simeq 1 + O(\epsilon^2) + 3^{1/2}\beta$, while $g_+(0)$ will not be changed in our approximation. Since the dominant contribution to the rates comes from the region $q^2 \simeq 0$, we predict for the branching ratio S, $S \equiv \Gamma(K_{Le3}{}^0)/2\Gamma(K_{e3}{}^+)$ ($S = 1.012$ if we consider only the phase-space difference[8]),

$$S=[1+O(\epsilon^2)]^2[1+O(\epsilon^2)+3^{1/2}\beta]^{-2}\times 1.012$$
$$=\{1-2(3)^{1/2}\beta[1+O(\epsilon^2)]^{-1}\}\times 1.012. \quad (10)$$

The latest world average[8] gives $\Gamma(K_{e3}^{\ +})=(3.93\pm 0.06)\times 10^6$ sec^{-1} and $\Gamma(K_{Le3}^{\ 0})=(7.22\pm 0.29)\times 10^6$ sec^{-1}, respectively. However, the error should be regarded with caution in view of internal disagreement in the data.[8] The above world averages yield $S=0.92\pm 0.04$. Thus there seems to be some indication of a sizable violation of the $|\Delta\vec{I}|=\frac{1}{2}$ rule and also of $S<1$. If we set $O(\epsilon^2)=0$ (by assuming strict asymptotic symmetry) in Eq. (10), the positive values of β, $\beta\simeq 0.02$, yield $S\simeq 0.94$, whereas the negative ones, $\beta\simeq -0.006$, give $S\simeq 1.02$. Since the $O(\epsilon^2)$ is likely to be negative,[7] its inclusion will enhance (but probably slightly) the effect obtained above. In the above estimate the effect of the usual radiative corrections involving the charged lepton is neglected. They have a smaller effect[9] than that of mixing if $\beta\simeq 0.02$. Thus we have seen that if the solutions $\beta\simeq 0.02$ are correct, a rather sizable violation of the $|\Delta\vec{I}|=\frac{1}{2}$ rule, $S\simeq 0.94$, is expected, and this is not in contradiction with the present experimental situation. If the $|\Delta\vec{I}|=\frac{1}{2}$ rule is well satisfied, we prefer the solutions $\beta\simeq -0.006$. Naively, we expect a similar trend also for the $K_{\mu 3}$ decay.[10] Our argument indicates that for the determination of the vector Cabibbo angle it is safer to use the $K_{Le3}^{\ 0}$ rate rather than the $K_{e3}^{\ +}$ rate, since g_+ is free from the SU(2) mixing effect. By using a form of $g_+(q^2)$, $g_+(q^2)=m_{K^{*0}}^{\ 2}(q^2+m_{K^{*0}}^{\ 2})^{-1}$, we obtain[11] $\sin\theta_V\simeq 0.209\pm 0.010$. This should be compared with the one determined[12] from β decay and μ decay, $\sin\theta_V=0.2095\pm 0.0086$.

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[3]This requires knowledge of the asymptotic behavior of $\delta u_{i\alpha}(\vec{k})$, which will be discussed elsewhere.

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