ANALYTIC HARD-PION CALCULATION OF THE $T=J=0$ $\pi\pi$ PHASE SHIFT*

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Using chiral SU(2) current algebra, we have derived a dynamical equation for the form factor which describes the sigma-field matrix element between single-pion states. With the aid of the model of Gell-Mann, Oakes, and Renner, we solve this equation in an effective-range approximation, and from unitarity deduce the $T=J=0$ $\pi\pi$ phase shift. Our analysis strongly favors the "up-down" set of phenomenological $\pi\pi$ phase shifts, and rules out a narrow ϵ resonance.

In a previous Letter[1] we showed how the concepts of analyticity and hard-pion current algebra, employed simultaneously, can form a basis for the dynamical calculation of a hadronic process.[2] We extend this approach here by deriving a dynamical equation for the sigma-field form factor, $F(t)$,

$$(4\omega_q\,\omega_p)^{1/2}\langle\pi^a(q)\,|\,\sigma(0)\,|\,\pi^b(p)\rangle = -\delta_{ab}\,F(t), \quad t=-(p-q)^2, \tag{1}$$

where the isoscalar[3] sigma field is defined by the following axial-vector-current, equal-time commutation relations:

$$\delta(x_0)[A_0{}^a(x), \partial_\mu A_\mu{}^b(0)] = -i\delta_{ab}\,\delta(x)\sigma(x), \tag{2a}$$

$$\delta(x_0)[A_0{}^a(x), \sigma(0)] = i\delta(x)\partial_\mu A_\mu{}^a(x), \tag{2b}$$

with $a, b = 1, 2, 3$ as isospin indices. Our aim is to get the $T=J=0$ $\pi\pi$ phase shift, δ_{00}. That the matrix element in Eq. (1) is related to it is clear from unitarity; the phase of $F(t)$ in the elastic $\pi\pi$ region is δ_{00}.

We begin by defining the off-shell three-point functions

$$\begin{aligned}
\delta_{ab}\,W_{\mu\nu}(q,p) &= \int dx dy\, e^{-iq\cdot x+ip\cdot y}\langle 0\,|\,TA_\mu{}^a(x)\sigma(0)A_\nu{}^b(y)\,|\,0\rangle,\\
\delta_{ab}\,W_\nu(q,p) &= \int dx dy\, e^{-iq\cdot x+ip\cdot y}\langle 0\,|\,T\partial_\mu A_\mu{}^a(x)\sigma(0)A_\nu{}^b(y)\,|\,0\rangle,\\
\delta_{ab}\,W(q,p) &= \int dx dy\, e^{-iq\cdot x+ip\cdot y}\langle 0\,|\,T\partial_\mu A_\mu{}^a(x)\sigma(0)\partial_\nu A_\nu{}^b(y)\,|\,0\rangle.
\end{aligned} \tag{3}$$

The last of these gives the off-shell form factor $F(q,p)$, extrapolated in the momenta q, p, and $k=p-q$. As specified in (2), the operator σ has the property that $\langle 0\,|\,\sigma(0)\,|\,0\rangle\neq 0$; so there are vacuum-state contributions, proportional to $\delta(k)$, in Eq. (3). In the process of applying the commutation relations to relate the W's, these vacuum contributions cancel; so we can consistently ignore them in using Eq. (3). With this understanding, we may express all three vertex functions in terms of analytic form factors by extracting pion poles:

$$W_{\mu\nu}(q,p) = F_{\mu\nu}(q,p) + \frac{F_\pi}{p^2+m_\pi{}^2}p_\nu F_\mu(p,q) + \frac{F_\pi}{q^2+m_\pi{}^2}q_\mu F_\nu(q,p) + \frac{F_\pi{}^2 q_\mu p_\nu}{(q^2+m_\pi{}^2)(p^2+m_\pi{}^2)}F(q,p), \tag{4a}$$

$$W_\nu(q,p) = \frac{-iF_\pi m_\pi{}^2}{q^2+m_\pi{}^2}\left[F_\nu(q,p) + \frac{F_\pi}{p^2+m_\pi{}^2}p_\nu F(q,p)\right], \tag{4b}$$

$$W(q,p) = [F_\pi{}^2 m_\pi{}^4/(q^2+m_\pi{}^2)(p^2+m_\pi{}^2)]\,F(q,p), \tag{4c}$$

where we take for the pion decay constant $F_\pi = 94$ MeV. Current-algebra commutation relations enable us to generate Ward identity relationships[4] among the vertex functions of Eq. (3). Expressed in terms of the form factors in Eqs. (4), the Ward identities imply the following off-shell dynamical equation:

$$F(q,p) = -m_\pi{}^2 + \frac{q_\mu p_\nu F_{\mu\nu}(q,p)}{F_\pi{}^2} + \frac{(q^2+p^2+m_\pi{}^2)}{F_\pi{}^2 m_\pi{}^2}\Delta^2(k), \tag{5}$$

where $\Delta^s(k)$ is the sigma-field propagator in the variable $t=-k^2$,

$$\Delta^s(k) = \int_{4m_\pi{}^2}^\infty \frac{\rho_s(x)}{x-t}dx, \tag{6}$$

with scalar spectral function ρ_s. If we take either q or p to the zero four-momentum limit, we regain the soft-pion equation first derived by Amatya, Pagnanenta, and Renner.[5] However, the generality of our hard-pion approach allows us to consider Eq. (5) on shell, $p^2=q^2=-m_\pi^2$, where we find

$$F(t)=-m_\pi^2+\frac{q_\mu p_\nu F_{\mu\nu}(t)}{F_\pi^2}-\frac{1}{F_\pi^2}\Delta^s(t). \tag{7}$$

Given the SU(2) current-algebra commutation relations and the defining relations (2a) and (2b), Eq. (7) is exact. It does not rely upon "strong" partially conserved axial-vector current,[6] but only upon the existence of a pion pole in the axial-current divergence. Moreover, no σ-particle pole is introduced; instead analyticity in t has been adhered to, so that the form factor $F(t)$ has the correct cut structure.

To solve Eq. (7), we shall approximate $q_\mu p_\nu F_{\mu\nu}(t)$ by using an analog of the "smoothness" hypothesis of Schnitzer and Weinberg.[4] This gives, in terms of a single unknown constant A, the relation[7]

$$\frac{q_\mu p_\nu F_{\mu\nu}(t)}{F_\pi^2}=A\frac{(t-2m_\pi^2)}{2m_A^2}\Delta^s(t). \tag{8}$$

The use of approximation (8) limits our study of Eq. (7) to the low-energy domain $|t|\lesssim 1\ \mathrm{BeV}^2$. Thus, we consider here only the $\pi\pi$ contribution to the spectral function $\rho_s(t)$,

$$\rho_s^{\pi\pi}(t)=\frac{3}{16\pi^2}\frac{P}{t^{1/2}}|F(t)|^2\theta(t-4m_\pi^2),\quad P^2=\tfrac{1}{4}(t-4m_\pi^2). \tag{9}$$

Although one may use Eqs. (7)-(9) as the basis for a phenomenological study of $T=J=0$ $\pi\pi$ phase shifts,[8] our calculation gains considerably in predictive power with the use of a second assumption—validity of the model of Gell-Mann, Oakes, and Renner.[9] Writing the Hamiltonian density as[9]

$$H=\overline{H}-u_0-cu_8, \tag{10}$$

where $\overline{H}$ is SU(3)⊗SU(3) invariant and the scalar densities u_0 and u_8 transform as components of $(\underline{3},\underline{3}^*)\oplus(\underline{3}^*,\underline{3})$, we infer from Eqs. (2a) and (10) that

$$\sigma(x)=-\tfrac{1}{3}(\sqrt{2}+c)[\sqrt{2}u_0(x)+u_8(x)]. \tag{11}$$

Using the Goldstone-Nambu interpretation of Ref. 9, which implies

$$c=2\sqrt{2}(m_\pi^2-m_K^2)/(m_\pi^2+2m_K^2), \tag{12}$$

we deduce that[10]

$$F(0)=-m_\pi^2. \tag{13}$$

Insertion of Eqs. (8) and (13) into Eq. (7) uniquely fixes the constant A, yielding $A=-m_A^2/(F_\pi m_\pi)^2$. The final form of the dynamical on-shell equation for $F(t)$ is thus

$$F(t)=-m_\pi^2-\frac{t}{2m_\pi^2F_\pi^2}\int_{4m_\pi^2}^{\infty}\frac{\rho_s(x)}{x-t}dx. \tag{14}$$

Equations (9) and (14) imply

$$\mathrm{Im}\,F(t)=-\frac{3}{64\pi}\frac{[t(t-4m_\pi^2)]^{1/2}}{m_\pi^2F_\pi^2}|F(t)|^2,\quad t\geq 4m_\pi^2. \tag{15}$$

Using the inverse-amplitude method of solution, with cutoff Λ, we find the following effective-range formula for F:

$$F(t)=-m_\pi^2\left[1-\frac{3}{64\pi^2}\frac{t}{F_\pi^2}\ln\frac{\Lambda}{m_\pi^2}+\frac{3}{32\pi}\frac{t^{1/2}}{F_\pi^2}P\left(\frac{2}{\pi}\ln\frac{2P+t^{1/2}}{2m_\pi}-i\right)\right]^{-1}, \tag{16}$$

containing a weak dependence on the single parameter Λ.

The most immediate use of Eq. (16) is a prediction of the $T=J=0$ $\pi\pi$ phase shift δ_{00} from the unitarity relation

$$\mathrm{Im}\,F=F^*e^{i\delta_{00}}\sin\delta_{00}. \tag{17}$$

We find that

$$\frac{P}{t^{1/2}}\cot\delta_{00}$$

$$=\frac{32\pi}{3}\frac{F_\pi^2}{t}-\frac{1}{2\pi}\ln\frac{\Lambda}{m_\pi^2}+\frac{P}{t^{1/2}}\frac{\pi}{2}\ln\frac{t^{1/2}+2P}{2m_\pi}. \quad (18)$$

The present experimental picture of $\pi\pi$ phase shifts is rather unclear.[11] An extensive series of experiments on various charge states of the production process $\pi N\rightarrow\pi\pi N$ have been performed, and the data from peripheral reactions have been extrapolated to yield the desired $\pi\pi$ phase shifts.[12-16] The usual extrapolation procedure contains a twofold ambiguity, $\delta_{00}\leftrightarrow\delta_{00}'=\delta_{11}-\delta_{00}+\frac{1}{2}\pi$, leading to four possible solutions since δ_{00} is near 90° at the ρ mass.[11] Recent analyses appear to favor either the "down-up"[13,14] or "up-down"[15,16] solutions, which correspond, respectively, to a sharp s-wave resonance or a very broad $\pi\pi$ enhancement. With this lack in understanding of even the qualitative nature of δ_{00}, it is clear that Eq. (18) can best be put to use by providing a solution to the "up-down" versus "down-up" impasse. We have therefore determined the constant Λ by making a least-squares fit to the phenomenological phase shifts of Ref. 13, which cover the energy range $625<t^{1/2}$ (MeV) <855. The result is shown in Fig. 1. The "up-down" fit, for which $\Lambda=350m_\pi^2$, is clearly preferred and for this case we plot the predicted δ_{00} from threshold to 900 MeV in Fig. 2. The phase shift passes through 90° at 960 MeV but does so too slowly to assign a meaningful width[17] [we get $(\frac{1}{2}d\delta/dt^{1/2})^{-1}\cong 3$ BeV]. Thus a single-level Breit-Wigner form of the $T=J=0$ $\pi\pi$ amplitude in the range $600\lesssim t^{1/2}$ (MeV) $\lesssim 1000$ is inappropriate for interpreting this very broad enhancement. Of course our analysis does not preclude the possibility of a fit to the data in terms of more than one resonance. The s-wave scattering length a is given by

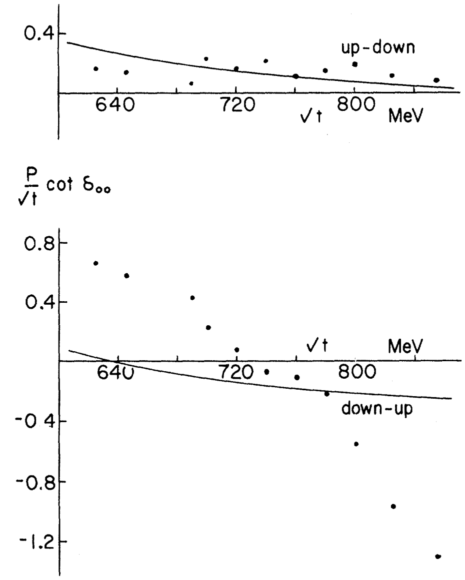

FIG. 1. Least-squares fit of Eq. (18) to the "up-down" and "down-up" solutions of Ref. 13. Data error bars, suppressed here, are shown in Fig. 2.

$$\frac{1}{a}=\frac{32\pi F_\pi^2}{6m_\pi}-\frac{m_\pi}{\pi}\ln\frac{\Lambda}{m_\pi^2}, \quad (19)$$

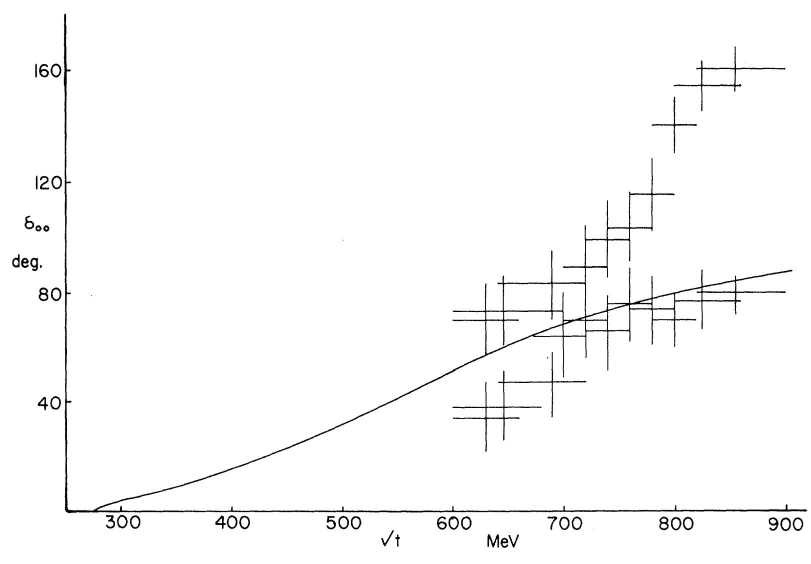

FIG. 2. Calculated phase shift, δ_{00}, obtained from the "up-down" fit. The data indicated are taken from Ref. 13.

to be compared with Weinberg's soft-pion results,[18]

$$1/a_W = 32\pi F_\pi^2/7m_\pi. \quad (20)$$

Numerically, we get $a = 0.17 m_\pi^{-1}$, essentially Weinberg's value.

We now summarize our findings. The main intent of this calculation has been to extend the analytic hard-pion approach introduced in Ref. 1 as a calculational tool in hadron dynamics. In doing so, we have been able, with a reasonable degree of certainty, to resolve the ambiguity inherent in phenomenological analyses of the $\pi\pi$ $T=0$ s wave. We find that the existence of a single narrow-width ϵ resonance is not likely to be understood within a calculation based on current algebra and elastic unitarity. Corollary to this is the conclusion that single-pole, zero-width dominance of the σ form factor is untenable. Further, we verify the Weinberg result for the s-wave scattering length, and in fact, agree rather closely with the low-energy phase shift which follows from a twice-subtracted dispersion relation.[19] By using the model of Gell-Mann, Oakes, and Renner to fix an unknown constant, and subsequently finding satisfactory agreement of our prediction, Eq. (18), with the "up-down" phase-shift solution (see Fig. 1), we indirectly give credence to their approach. Since the comparison of their model with experiment is still a matter of some uncertainty,[6,20] it is useful to be able to submit our results to what should be as broad a spectrum of experimental comparisons as possible.

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[3]That the sigma field defined in Eqs. (2a) and (2b) is an isoscalar quantity to a good approximation has been empirically verified by L. J. Gutay, F. T. Meiere, and J. H. Scharenguivel, Phys. Rev. Lett. 23, 431 (1969); M. G. Olsson and L. Turner, Phys. Rev. Lett. 20, 1127 (1968).

[4]The significance of Ward-identity relations in hard-pion calculations was first pointed out by H. J. Schnitzer and S. Weinberg, Phys. Rev. 164, 1828 (1967).

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[7]Off shell, one can write in general

$$F_{\mu\nu}(q,p) = g_A^{-2}\Delta_{\mu\eta}^A(q)\Delta_{\nu\rho}^A(p)\Delta^s(k)\Gamma_{\eta\rho}(q,p),$$

where Δ^A and Δ^s are axial-vector and scalar propagators and $\Gamma_{\eta\rho}$ is a proper vertex part. With the smoothness assumption, we take $\Gamma_{\eta\rho}(q,p) = A\delta_{\eta\rho}$, $A = \text{const}$. Thus, the t-plane cut of $F_{\mu\nu}(t)$ is entirely contained in $\Delta^s(t)$. In Eq. (8) we have used $q_\mu\Delta_{\mu\eta}^A(q) = q_\eta g_A^2/m_A^2 = q_\eta F_\pi^2$.

[8]Comparison with the data may be made in terms of the two parameters Λ and A [Λ is defined Eq. (15)]. Even with this added freedom we find our equation rather inconsistent with a narrow ϵ, at least in the $\pi\pi$ truncation of ρ_s.

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[10]See also H. Suura, Phys. Rev. Lett. 23, 551 (1969).

[11]For example, see the discussion by J. D. Jackson, Rev. Mod. Phys. 42, 12 (1970).

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[13]J. H. Scharenguivel *et al.*, Phys. Rev. 186, 1387 (1969).

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[17]This general conclusion is confirmed in a method for unitarizing Veneziano partial waves: E. P. Tryon, "Veneziano Versus Partial-Wave Dispersion Relations: A Unitary, Crossing-Symmetric Synthesis and the $\pi\pi$ Phase Shifts" (to be published). We thank Dr. Tryon for a correspondence.

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[19]E. P. Tryon, Phys. Rev. Lett. 20, 769 (1967). Since Tryon uses current-algebra scattering lengths as *input*, this agreement is not too surprising.

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