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SCALING BEHAVIOR IN $pp \to \pi +$ ANYTHING AT HIGH ENERGY*

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We present evidence that the inclusive reaction $p+p \to \pi +$ anything approaches a scaling limit at accelerator energies.

Although the bulk of high-energy reactions is inelastic, the variety and complexity of final states encountered have made it difficult for the experimentalist to decide what to measure and have prevented detailed theoretical analysis of these processes. Of all inelastic processes, perhaps the most amenable to experimental study and to theoretical treatment are inclusive reactions—reactions in which some final particle or property of the final state is studied irrespectively of whatever else is happening. As an example of such a reaction, we discuss here the process $p+p \to \pi +$anything.

The differential cross section for this process depends in general on three variables. We can conveniently take these as the square of the center-of-mass energy s, and the longitudinal and transverse components of the momentum of the outgoing pion in the c.m. system,

$$d\sigma_\pi = \frac{d^3p}{p^0} f(p_\perp, p_\parallel, s) = \frac{\pi dp_\perp{}^2 dp_\parallel}{(p_\perp{}^2 + p_\parallel{}^2 + \mu^2)^{1/2}} f(p_\perp, p_\parallel, s). \qquad (1)$$

On the basis of heuristic arguments, Feynman[1] has suggested that the structure function $f(p_\perp, p_\parallel, s)$ scales at high energy. That is, as $s \to \infty$ it becomes only a function of $p_\perp$ and the ratio $x = 2p_\parallel/s^{1/2}$. Thus,

$$d\sigma_\pi \xrightarrow[s\to\infty]{} \frac{\pi dp_\perp{}^2 dx}{[x^2 + 4(p_\perp{}^2 + \mu^2)/s]^{1/2}} f(x, p_\perp). \qquad (2)$$

Analogous expressions have been suggested by Amati, Stanghellini, and Fubini,[2] and Wilson,[8] on the basis of the multiperipheral model.

The integrated inclusive cross section counts the production of n pions n times over. Hence $\Sigma_\pi \equiv \int d\sigma_\pi = \Sigma n\sigma_n$, where σ_n is the cross section for the production of n pions. It is related to the average number of π's produced in inelastic collisions

$$\bar{n}_\pi = \frac{\Sigma n\sigma_n}{\Sigma \sigma_n} = \frac{\Sigma_\pi}{\sigma_{\text{tot inel}}}. \qquad (3)$$

If the total inelastic cross section for pp scattering approaches a constant as s goes to infinity, which we shall assume, and if Feynman's scaling holds true with $f(0, p_\perp) \neq 0$, then it follows that the average number of pions produced in pp collisions increases logarithmically with s,[4]

$$\bar{n}_{\pi^\pm} = \left[\frac{\pi}{\sigma_{\text{tot inel}}} \int_0^\infty dp_\perp{}^2 f^\pm(0, p_\perp)\right] \ln(s) + \text{const}^\pm = c^\pm \ln(E/m) + d^\pm, \qquad (4)$$

where E is the incident energy in the laboratory frame. Recent cosmic-ray experiments[5] with a hydrogen target, including energies up to 800 GeV, clearly show such a logarithmic growth of the average multiplicity.

In this note we investigate the extent to which existing accelerator data for the process $p+p \to \pi +$anything have approached the Feynman scaling limit. In particular, if these data are already asymptotic, then the coefficient of $\ln(s)$ in Eq. (4) calculated from the accelerator data must agree with the coefficient of $\ln(s)$ determined experimentally from the presumably asymptotic cosmic-ray data.

The existing accelerator data[6-8] cannot be used directly to compute the integral indicated in Eq. (4) since no data for $x = 0$ and varying $p_\perp$ exist. We can, however, fit the existing data with a factorized form

$$f^\pm(x, p_\perp) = F(p_\perp) G^\pm(x), \qquad (5)$$

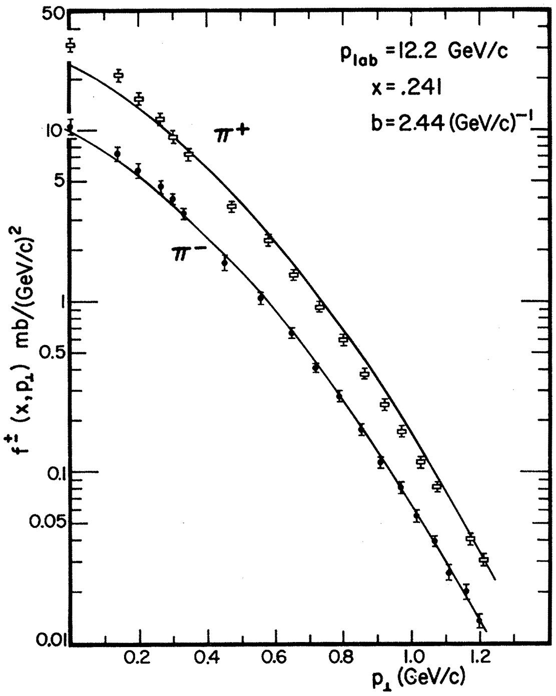

FIG. 1. Fit to the $p_\perp$ dependence at fixed x using Eqs. (5)-(6). The data points are from Ref. 6.

and use this form to extrapolate into the $x \simeq 0$ region.[9] We have determined $F(p_\perp)$ from the π^- spectra at fixed x of Ref. 6. We find that the functional form

$$F(p_\perp) = \exp\{-b(p_\perp + p_\perp^2/m)\}, \tag{6}$$

where m is the nucleon mass, provides an excellent fit to these data as shown in Fig. 1. We have chosen to fit the π^- data because, for reasons to be discussed below, we believe that these data are more nearly asymptotic than the π^+ data. However, as can be seen from Fig. 1, this parametrization is also adequate in the case of positive pions. In Fig. 2 we display the experimental values obtained for $G^\pm(x)$ by using $F(p_\perp)$ as given in Eq. (6). We observe that indeed points of different $p_\perp$, to a reasonable approximation, fall on the same curve, and this verifies the factorized form[10] of Eq. (5). The general behavior of the functions $G^\pm(x)$ as shown in Fig. 2

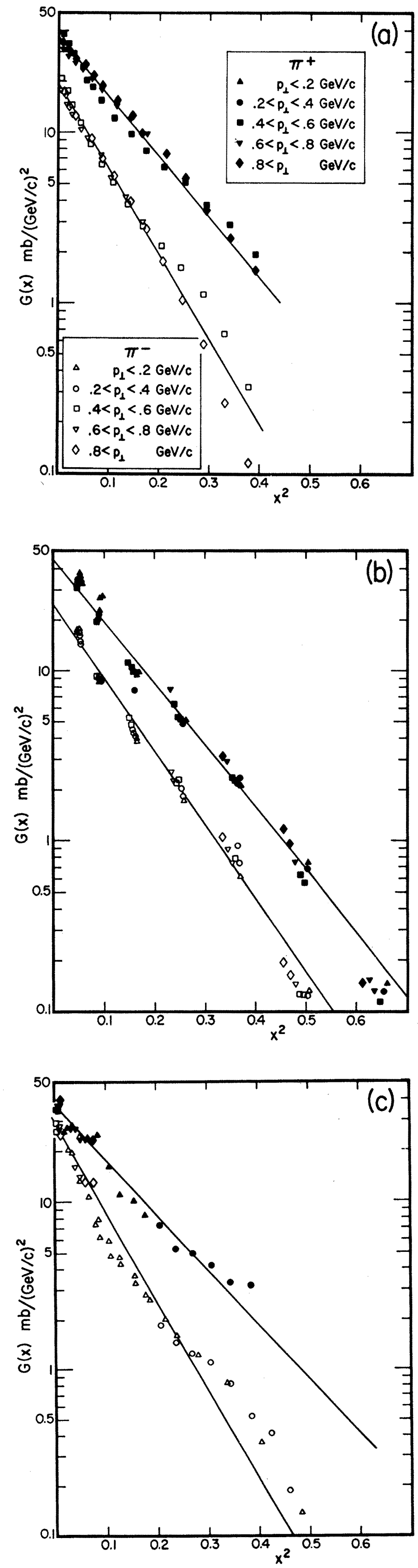

FIG. 2. The x dependence of the reduced data. (a) $p_{\rm lab} = 12.2$ GeV/c, $G_0^+ = 36$ mb/GeV^2, $G_0^- = 19$ mb/GeV^2, $a^+ = 8.2$, $a^- = 11.5$ (data from Ref. 6). (b) $p_{\rm lab} = 19.2$ GeV/c, $G_0^+ = 44$ mb/GeV^2, $G_0^- = 25$ mb/GeV^2, $a^+ = 8.5$, $a^- = 10$ (data from Ref. 7). (c) $p_{\rm lab} = 30.0$ GeV/c, $G_0^+ = 37$ mb/GeV^2, $G_0^- = 29$ mb/GeV^2, $a^+ = 7.4$, $a^- = 12.1$ (data from Ref. 8).

suggests the parametrization

$$G^{\pm}(x) = G_0^{\pm} \exp\{-a^{\pm}x^2\}, \tag{7}$$

although the data oscillate a fair amount about this fit. For our purposes, as we are mainly interested in the point $x=0$, the straight-line fit of Fig. 2 suffices.

With the parametrizations of Eqs. (5)-(7), it is now straightforward to calculate the coefficient of $\ln(s)$ from the accelerator data.[11] Using the value $\sigma_{\text{tot inel}} = 30.5$ mb, we obtain the results for π^- and π^+ that are shown in Table I. The coefficients obtained for the π^- show a rise with energy, while the coefficients of the π^+ are more stable and larger than c^-. In order to conserve charge in the asymptotic limit, c^- and c^+ must be equal so that the trend for c^- and c^+ to approach each other with increasing energy is a tendency in the right direction. The rise in c^- means that the data at lower energy are approaching the Feynman limit from below. The near constancy of c^+ reflects the combination of competing physical effects. Part of c^+ comes from the same kind of processes that contribute to c^-, and this part rises with energy to a limiting value. The rest of c^+ comes from contributions to the π^+ spectrum which must eventually vanish at $x=0$, and this remainder decreases with energy. Such contributions occur where the final pion can carry the incident charge, and they also are expected to scale at high energy.

A least-squares fit to the cosmic-ray data of Ref. 5 gives the average number of charged particles as

$$\bar{n}^{\text{ch}} = 0.89 \ln(E/m) + 1.10. \tag{8}$$

Assuming that besides pions the only other significant production is K production, and that the K/π ratio[5,12] is about $\frac{1}{5}$, we obtain

$$c_{\exp}^{-} = c_{\exp}^{+} = 0.36. \tag{9}$$

The agreement of these numbers with those in Table I constitutes good evidence that the process $pp \to \pi + \text{anything}$ is approaching the scaling limit at $x=0$ at 30 GeV/c.

It is also possible to integrate the functional form of Eqs. (5)-(7) and calculate the constant term in $\bar{n}$. Using the values of the 30-GeV/c parameters b and $a^{\pm}$ indicated in Figs. 1 and 2(c), we obtain

$$\bar{n}^- = 0.37 \ln(E/m) - 0.19, \tag{10a}$$

$$\bar{n}^+ = 0.47 \ln(E/m) - 0.02. \tag{10b}$$

We add these two numbers, add 20% for K production, and finally add 1.5 for the average number of final protons [obtained by integrating the parametrization of the proton spectrum at 30 GeV/c given in Eq. (1) of Ref. 8]. This gives a total charged multiplicity

$$\bar{n}^{\text{ch}} = 1.01 \ln(E/m) + 1.3, \tag{11}$$

in good agreement with the cosmic-ray fit of Eq. (8). This suggests that the asymptotic regime has not only been reached at 30 GeV/c for $x=0$, as discussed above, but also for those small values of x which contribute the bulk of the cross section.[13]

We remark that the scaling hypothesis implies a weak energy dependence of the average value of $p_\perp$:

$$\langle p_\perp \rangle_\pi = \frac{\int p_\perp d\sigma_\pi}{\Sigma_\pi} = \langle p_\perp \rangle_\infty \left\{1 - \frac{\lambda}{\ln(E/m)}\right\}. \tag{12}$$

With our parametrization of the accelerator data, we find[14] the asymptotic value $\langle p_\perp \rangle_\infty = 390$ MeV/c and the coefficient $\lambda = 0.58$. This logarithmic correction is a small effect since, for example, at 30 GeV/c $\langle p_\perp \rangle = 325$ MeV/c.

The average pionic inelasticity ρ_π, defined as the ratio of the total energy that goes into pions to the total available energy, is constant if the structure function scales.[2] With our parametrization we find

$$\rho_{\pi\pm} = \frac{\int d^3p\, f(p_\perp, x)}{s^{1/2}\sigma_{\text{tot inel}}} = c^{\pm}\left(\frac{\pi}{4a^{\pm}}\right)^{1/2}, \tag{13}$$

which, using the 30-GeV/c parameters, gives $\rho_{\pi^+} = 0.155$, $\rho_{\pi^-} = 0.095$. Since the average longitudinal momentum carried away by the pions vanishes, the inelasticity, ρ_π is not changed by a Lorentz transformation to the laboratory frame If we multiply the charged pionic inelasticity by $\frac{3}{2}$ to account for energy taken away by π^0, we arrive at a figure of about 40% for the total energy carried away by pions, in good agreement with cosmic-ray results.[12]

In summary, we have shown evidence that Feynman's limit is closer than one might otherwise have expected, and pointed out some sim-

Table I. Coefficient of $\ln(s)$ from accelerator data.

Incident lab. momentum (GeV/c)	c^-	c^+
12.2	0.24	0.46
19.2	0.32	0.56
30	0.37	0.47

ple experimental consequences that follow from the validity of the scaling limit. We should emphasize that we have relied heavily on the cosmic-ray data of Ref. 5, since they are the only data obtained from a hydrogen target. Other cosmic-ray data can differ by as much as a factor of 2 from that of Ref. 5. The scaling limit will of course be tested at the accelerators now under construction, but even with the ones available at this date it is possible to sharpen the evidence for this limit. A good measurement of the cross section at $x=0$ for various values of $p_\perp$ would provide the coefficients c^+ and c^- directly, independently of any extrapolation procedures. We urge that such an experiment be done in the near future.

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[1]R. P. Feynman Phys. Rev. Lett. 23, 1415 (1969). See also, J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, Phys. Rev. 188, 2159 (1969).

[2]D. Amati, A. Stanghellini, and S. Fubini, Nuovo Cimento 26, 896 (1962).

[3]K. G. Wilson, Acta Phys. Austr. 17, 37 (1963).

[4]The convergence of the integral in Eq. (4) is assumed. This requires a damping of $f(0,p)$ at large values of $p_\perp$, which is a well-known experimental fact.

[5]L. W. Jones, in Proceedings of the International Conference on Expectations for Particle Reactions at New Accelerators, University of Wisconsin, April 1970 (to be published); K. N. Erickson, thesis, University of Michigan, 1970 (unpublished).

[6]J. L. Day *et al.*, Phys. Rev. Lett. 23, 1055 (1969).

[7]J. W. Allaby *et al.*, in Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, Austria, September 1968 (to be published).

[8]E. W. Anderson *et al.*, Phys. Rev. Lett. 19 198 (1967).

[9]Similar factorization of the structure function has been discussed by Day *et al.* (Ref. 6) and by S. D. Drell, in Proceedings of the International Conference on Expectations for Particle Reactions at the New Accelerators, University of Wisconsin, April 1970 (to be published).

[10]From the data of Ref. 8 we have excluded the points at $p_\perp > 1$ GeV/c. Beyond this value, the experimental points seem to be somewhat below our parametrization of Eq. (8). These points, however, are down by a factor of 100 and do not affect the integrated results.

[11]We estimate that the uncertainties in extrapolation of the existing data and the uncertainties in normalization which exist in the present experiments do not allow us to compute the coefficients $c^\pm$ to better than an accuracy of 15 %.

[12]F. Turkot, in Proceedings of the Topical Conference on High-Energy Collisions of Hadrons, CERN, Geneva, Switzerland, January 1968(CERN Report No. 68-72, 1968).

[13]Some data from Serpukhov on an Al target at 70 GeV/c exist. We have not included these data in our analysis because they consist only of large $-x$ points which do not allow a reliable extrapolation to $x=0$ and because of the uncertainties of nuclear corrections.

[14]It should be noted that with our functional form $\langle p_\perp\rangle_\infty$ and λ do not depend on the values of the slope parameters $a^\pm$. Hence, $\langle p_\perp\rangle$ is the same for π^+ and π^-.