GENERALIZED DIFFUSION EQUATION FOR SUPERCONDUCTING ALLOYS*

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Eilenberger's transportlike equations for a superconductor of type II can be simplified very much in the dirty limit. In this limit a diffusionlike equation is derived which is the generalization of the de Gennes–Maki theory for dirty superconductors to arbitrary values of the order parameter.

Recently Eilenberger[1] derived transportlike equations for superconductors of type II which, for a small order parameter $\Delta(\vec{r})$, reduce to the Boltzmann equation introduced and studied by Lüders.[2] These transportlike equations are much easier to handle than the original Gor'kov[3] equations since the number of variables is reduced.

In this paper we study Eilenberger's equations for a dirty superconductor. We expect that a short mean-free path makes the theory simple since the motion of the electrons is always nearly isotropic. Indeed, for small mean-free paths, Lüders' Boltzmann equation reduces to the simple diffusion equation introduced by de Gennes.[4] Here we extend this theory to the general nonlinear case. We derive a diffusionlike equation from which all information about a dirty superconductor can be obtained. This equation expanded in powers of the order parameter up to terms of the order Δ^3 gives Maki's[5] dirty-limit theory including the corrections derived by Caroli, Cyrot, and de Gennes.[6] However, our diffusion equation is valid for all values of Δ.

We start with a collection of some equations derived by Eilenberger.[1] Eilenberger introduces the functions $g(\omega, \vec{r}, \vec{v})$ and $f(\omega, \vec{r}, \vec{v})$ which are closely related to the Green's functions of a superconductor. Indeed, these quantities are the impurity-averaged Green's functions integrated over the energy variable. They are solutions of the following transportlike equation:

$$\{2\omega + \vec{v}\cdot\hat{\partial}\} f(\omega, \vec{r}, \vec{v}) = 2\Delta g(\omega, \vec{r}, \vec{v}) + nv\oint d\Omega_{\vec{v}'} \frac{d\sigma(\vec{v}, \vec{v}')}{d\Omega} \{g(\omega, \vec{r}, \vec{v}) f(\omega, \vec{r}, \vec{v}') - f(\omega, \vec{r}, \vec{v}) g(\omega, \vec{r}, \vec{v}')\}. \tag{1}$$

We restrict ourselves to spherical Fermi surfaces. $\vec{v}$ denotes the Fermi velocity and $\hat{\partial}$ is defined by $\hat{\partial} = \vec{\partial} + 2ie\vec{A}(\vec{r})$, with $\vec{A}(\vec{r})$ a vector potential of the magnetic field. We can assume $\omega = (2n+1)\pi T > 0$. The other symbols in Eq. (1) have their standard meanings. For more details, the reader is referred to Eilenberger's paper.[1] The functions f and g are connected by

$$g(\omega, \vec{r}, \vec{v}) = [1 - f(\omega, \vec{v}, \vec{r}) f^*(\omega, -\vec{v}, \vec{r})]^{1/2}. \tag{2}$$

These equations are completed by the self-consistency conditions

$$\Delta(\vec{r}) \ln\frac{T}{T_c} + 2\pi T \sum_{\omega > 0} \left[\frac{\Delta(\vec{r})}{\omega} - \oint \frac{d\Omega_{\vec{v}}}{4\pi} f(\omega, \vec{r}, \vec{v})\right] = 0, \tag{3}$$

$$\vec{j}_s(\vec{r}) = \frac{1}{4\pi}\nabla \times [\vec{B}(\vec{r}) - \vec{B}_e(\vec{r})] = 2ieN(0)2\pi T \sum_{\omega > 0} \oint \frac{d\Omega_{\vec{v}}}{4\pi} \vec{v} g(\omega, \vec{r}, \vec{v}). \tag{4}$$

T_c denotes the transition temperature of the bulk superconductor without magnetic field.

For a dirty superconductor one expects that the functions $f(\omega, \vec{r}, \vec{v})$ and $g(\omega, \vec{r}, \vec{v})$ become nearly isotropic with respect to $\vec{v}$. Indeed, it was shown by Lüders[2] that in the limit $\Delta \to 0$ this assumption leads directly to the diffusion equation for de Gennes' kernel of the linearized self-consistency equation. Therefore we apply Lüders' method to Eq. (1), expanding $f(\omega, \vec{r}, \vec{v})$:

$$f(\omega, \vec{r}, \vec{v}) = F(\omega, \vec{r}) + \hat{v}\cdot\vec{F}(\omega, \vec{r}). \tag{5}$$

Here we neglect higher order terms and we have defined $\hat{v} = \vec{v}v^{-1}$. Using Eqs. (2) and (5) we get to lowest order for $g(\omega, \vec{r}, \vec{v})$:

$$g(\omega, \vec{r}, \vec{v}) = G(\omega, \vec{r}) + \hat{v}\cdot\vec{G}(\omega, \vec{r}), \tag{6}$$

where $G(\omega, \vec{r})$ is defined by

$$G(\omega, r) = [1 - |F(\omega, \vec{r})|^2]^{1/2}, \tag{7}$$

and $\vec{G}(\omega, \vec{r})$, by

$$\vec{G}(\omega, \vec{r}) = \frac{1}{2} \frac{F(\omega, \vec{r})\vec{F}^*(\omega, \vec{r}) - F^*(\omega, \vec{r})\vec{F}(\omega, \vec{r})}{G(\omega, \vec{r})}. \tag{8}$$

The above approximation of $g(\omega, \vec{r}, \vec{v})$ seems to be sufficient.

Now it is easy to get a set of equations for the "densities" $F(\omega, r)$, $G(\omega, \vec{r})$ and the "currents" $\vec{F}(\omega, \vec{r})$, $\vec{G}(\omega, \vec{r})$. Integrating Eq. (1) over all directions of the Fermi velocity we get

$$2\omega F(\omega, \vec{r}) + \tfrac{1}{3}\vec{v}\hat{\partial}\vec{F}(\omega, \vec{r}) = 2\Delta G(\omega, \vec{r}), \tag{9}$$

and multiplying Eq. (1) first by $\hat{v}$ and then integrating over $\hat{v}$, we find

$$2\omega\vec{F}(\omega, \vec{r}) + v\hat{\partial}F(\omega, \vec{r}) = 2\Delta\vec{G}(\omega, \vec{r}) + \tau_{tr}^{-1}[\vec{G}(\vec{r}, \vec{v})F(\vec{r}, \vec{v}) - \vec{F}(\vec{r}, \vec{v})G(\vec{r}, \vec{v})], \tag{10}$$

where τ_{tr} is the transport lifetime.

With the help of Eq. (8) we can eliminate the currents $\vec{F}$ and $\vec{G}$ from Eqs. (9) and (10). As a result of this simple calculation we obtain an equation for $F(\omega, \vec{r})$:

$$2\omega F(\omega, \vec{r}) - D\hat{\partial}\left[G(\omega, \vec{r})\hat{\partial}F(\omega, \vec{r}) + \frac{1}{2}\frac{F(\omega, \vec{r})}{G(\omega, \vec{r})}\hat{\partial}|F(\omega, \vec{r})|^2\right] = 2\Delta(\vec{r})G(\omega, \vec{r}). \tag{11}$$

$G(\omega, \vec{r})$ is given by Eq. (7) and we have defined $D = \frac{1}{3}\tau_{tr}v^2$.

Equation (11) is a generalization of de Gennes' diffusion equation to arbitrary values of Δ. Of course, this relation is only valid in the dirty limit. Precisely speaking, in deriving Eq. (11) we made the following assumptions:

$$G \gg 2\tau_{tr}\omega, \quad F \gg 2\tau_{tr}\Delta, \quad |F| \gg |\hat{v}\cdot\vec{F}|. \tag{12}$$

In applying Eq. (11), the restrictions (12) should always be checked. For simple situations it can be shown that these restrictions are equivalent to the dirty-limit condition

$$2\pi T_c \tau_{tr} \ll 1. \tag{13}$$

Many known results can be derived quite easily from Eq. (11). In particular, an expansion of this equation in terms of Δ up to the term Δ^3 gives Maki's dirty-limit theory including the correction term of Caroli, Cyrot, and de Gennes.

At surfaces and interfaces Eq. (11) has to be completed by boundary conditions. In the linear case the boundary condition at the metal-insulator interface reads

$$\vec{n}\cdot\hat{\partial}F(\omega, \vec{r}) = 0, \tag{14}$$

where $\vec{n}$ is normal to the interface.

We expect Eq. (14) to be valid also in the general nonlinear case. So far we were able to prove this relation only for specular reflection.

Finally we write down the self-consistency conditions in terms of $F(\omega, \vec{r})$. For the order parameter we get, of course,

$$\Delta(\vec{r})\ln\frac{T}{T_c} + 2\pi T\sum_{\omega>0}\left[\frac{\Delta(\vec{r})}{\omega} - F(\omega, \vec{r})\right] = 0 \tag{15}$$

and for the current density, using Eqs. (6), (8), and (10) we get

$$\vec{j}_s(\vec{r}) = 2ieN(0)\pi TD\sum_{\omega>0}[F^*(\omega, \vec{r})\hat{\partial}F(\omega, \vec{r}) - F(\omega, \vec{r})(\hat{\partial}F(\omega, \vec{r}))^*]. \tag{16}$$

Of course, other quantities like the free energy or the density of states can be expressed quite easily in terms of $F(\omega, \vec{r})$.

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ELECTRONIC TRANSPORT IN AMORPHOUS SILICON FILMS

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Drift mobility and conductivity measurements were made between 290 and 85°K on amorphous silicon specimens prepared by glow-discharge decomposition of silane. The results suggest that excess electrons drift in the extended states with a mobility of about 10 $cm^2\ sec^{-1}\ V^{-1}$. At lower temperatures, phonon-assisted hopping occurs through localized states occupying a range of 0.2 eV below the extended states. Conductivity results also suggest hopping transport near the Fermi energy.

During recent years a model of the electronic structure of amorphous materials has evolved, largely through the work of Mott,[1-4] Davis and Shaw,[5] Cohen, Fritzsche, and Ovshinsky,[6] and Cohen.[7,8] Although its general features appear to be in agreement with experimental evidence, there still exists a serious lack of experimental information whereby some of the more detailed predictions could be tested, particularly those connected with electronic transport properties.

In the following, we should like to report drift mobility measurements in amorphous silicon which, in conjunction with conductivity measurements on the same specimens, allowed us to make a number of meaningful deductions about electronic states and the transport mechanism.

The specimens used in the present work consisted essentially of an amorphous silicon film, about 1 μm thick, sandwiched between two evaporated aluminium electrodes. The amorphous layer was produced by an electrodeless rf glow discharge in silane gas at a substrate temperature of about 200°C. As shown by the work of Chittick, Alexander, and Sterling[9] and Chittick,[10] a distinctive feature of specimens prepared by this technique is their high resistivity, appreciably larger than that of specimens prepared by vacuum evaporation[11] or sputtering.[12]

In the drift mobility measurements, we used pulsed electron-beam excitation (10-nsec pulses, 6-keV electrons) to generate excess carriers near the top electrode. The transit of electrons could be observed in a pulsed applied field, synchronized to the excitation. So far, no hole signals have been detected. Further details of this experimental technique can be found in the work of Spear.[13] Conductivity measurements were carried out with steady fields of up to 40 kV cm^{-1}. Specimens were Ohmic in this range, but some asymmetry was apparent on field reversal.

The five specimens so far investigated[14] led to results showing the same general features. Except at temperatures below 200°K, the reproducibility was good. The following discussion

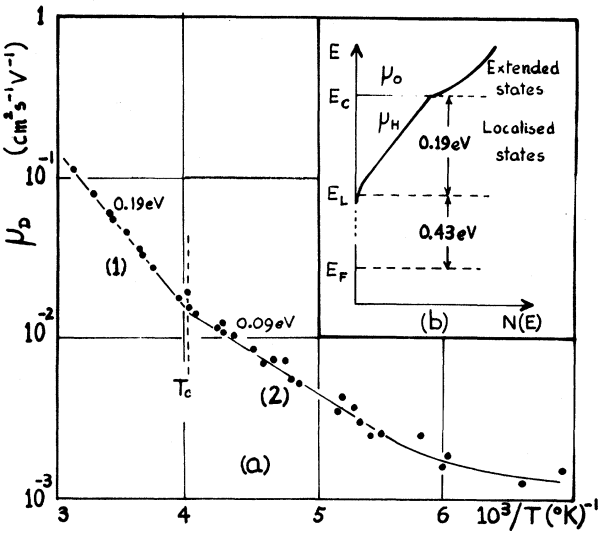

FIG. 1. (a) Temperature dependence of the electron drift mobility μ_D in an amorphous silicon film, 1.3 μm thick. (b) Model of the electronic-state distribution in an amorphous solid. The linear distribution of localized states has been chosen for illustration and does not necessarily follow from the results described.