S-MATRIX ANALYSIS OF K_{l3} FORM FACTORS: ZERO OF THE SCALAR FORM FACTOR AND THE PARAMETER ξ *

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We present an S-matrix calculation of the K_{l3} form factors which can explain the parameter $\xi \simeq -1$ of the more recent polarization measurements. Existence of a zero in the scalar form factor near the K_{l3} physical region and below the $K\pi$ threshold plays an important role. We show further that the theoretical arguments based on the soft-meson theorems are inadequate to predict such a zero in the scalar form factor irrespective of the symmetry-breaking patterns employed. Implications of the zero to the off-mass-shell extrapolation of the form factors are briefly discussed.

It has been recognized for some time that[1] the ξ parameters determined from a study of the muon polarization in $K_{\mu 3}$ decay are much larger in magnitude than those from a study of the $K_{\mu 3}/K_{e3}$ branching ratio. This difference has raised various speculations on our present understandings of the weak interaction theory, such as μ-e universality in strangeness-changing decays. But a more recent branching-ratio experiment[2] gives a result for ξ compatible with the results from polarization measurements[3] and in agreement with μ-e universality. Although the matter is not settled yet, there seems to be a growing tendency among the experts in this field in favor of the polarization-study value $\xi \simeq -1$, since the muon polarization study measures ξ directly and does not need the same experimental complication concerning detection efficiency as in the branching-ratio measurements.

The purpose of this note is to report on a simple interesting S-matrix analysis in which a zero of the form factor between $(m_K - m_\pi)^2$ and $(m_K + m_\pi)^2$ can explain $\xi \simeq -1$. To our knowledge, no earlier calculations in this area[4] have noted the possibility of a form-factor zero. Our argument is independent of any assumption of the off-mass-shell extrapolations as we use only the on-mass-shell S-matrix quantities. We show further that the theoretical arguments based on the soft-meson theorems of the SU(3)$\otimes$SU(3) current algebras are not sufficient to conclude $\xi \simeq -1$, contrary to the findings of Berman and Roy.[5] We find in addition that the theoretical discussions making use of the pole-dominance version of partial conservation of axial-vector current and an approximate SU(3) symmetry are not sufficient either to argue $\xi = -1$ in contrast with the work of Brandt and Preparata.[6] Instead, we find that such soft-meson theories tell us simply the well-known Ademollo-Gatto theorem[7] and that we need to know the πK scattering amplitudes beyond those determined by the Weinberg linear expansion[8] incorporating the low-energy theorems and Adler's condition.[9] Clearly, the Weinberg technique will introduce many more parameters even when the next higher order terms are included, and the low-energy theorems and Adler's condition are not sufficient to determine them all.

The starting point of our discussion is to observe the existence of a zero in the scalar form factor $F(s)$ [defined by Eq. (2) below] somewhere below the physical threshold of πK scattering but above the K_{l3} physical region. The existence of a zero in the nonet pseudoscalar scattering amplitudes has been speculated to be possibly a general common feature.[10] Indeed a zero exists in the πK scattering amplitude determined by the Weinberg linear expansions incorporating the low-energy theorems, Adler's condition, and a $(\underline{3}, \underline{3}^*) \oplus (\underline{3}^*, \underline{3})$ form of SU(3)$\otimes$SU(3) breaking theory.[12] However, we know of no convincing theoretical arguments based on the low-energy theorems that predict existence of a zero in the form factors. Poles exist in the form factor if there are single-particle states which can contribute to it but the arbitrariness associated with possible zeros is subject to the experiments as the form factor is an observable quantity. We show in the following that the experimental results $\xi \simeq -1$ imply a zero in $F(s)$.

Before we discuss the zero of $F(s)$, we note what one gets from the theoretical arguments based on the soft-meson theorems of the SU(3) $\otimes$ SU(3) current algebras. Let us define, following the usual convention, the K_{l3} form factors $f_\pm(s)$ by the hadronic matrix element:

$$(4k_0 p_0)^{1/2} \langle \pi^0(k) | V_\mu^{(K)}(0) | K^+(p) \rangle = 2^{-1/2} [f_+(s)(p+k)_\mu + f_-(s)(p-k)_\mu], \tag{1}$$

where $s=(p-k)^2$. Then we have

$$(4k_0p_0)^{1/2}\langle\pi^0(k)|\partial_\mu V_\mu^{(K)}(0)|K^+(p)\rangle=(i/\sqrt{2})[f_+(s)(m_K^2-m_\pi^2)+f_-(s)s]\equiv(i/\sqrt{2})F(s), \quad (2)$$

where $F(s)$ is the $I=\frac{1}{2}$ s-wave $K\pi$ form factor. Using the equal-time commutation relations for the integrated charge densities of SU(3)⊗SU(3) and the Heisenberg equations of motion, one can show that[13] $F(s)$ is related to the matrix elements of the σ terms,

$$-i\int d^3y[A_0^\alpha(\vec{y},x_0),\partial_\mu A_\mu^\beta(x)]\equiv\sigma_{\alpha\beta}(x),$$

by

$$F(s)=-(8p_0k_0)^{1/2}\langle 0|\sigma_{\pi^-K^+}(0)-\sigma_{K^+\pi^-}(0)|K^+(p)\pi^-(k)\rangle. \quad (3)$$

The matrix element of $\sigma_{\pi K^+}$ ($\sigma_{K^+\pi^-}$) can be obtained after the usual manipulation from the off-mass-shell scattering amplitude $M_{K^+\pi^-}(s,t,u;p^2,k^2,p'^2,k'^2)$ for $K^+(p)+\pi^-(k)\to K^+(p')+\pi^-(k')$ in the soft limit $k'\to 0$ ($p'\to 0$). If one uses the results of Griffith[11] for $M_{K^+\pi^-}$ which incorporate the $(\underline{3},\underline{3}^*)\oplus(\underline{3}^*,\underline{3})$ breaking theory, one then obtains $F(s)=m_K^2-m_\pi^2$ so that $f_+(s)=1$ and $f_-(s)=0$ in agreement with the Ademollo-Gatto theorem. This implies that the theoretical arguments based on the soft-meson theorems and the usual statements for the σ terms which are a measure of chiral-symmetry breaking are not sufficient to give $\xi\cong-1$, while they may be consistent with $\xi\cong 0$. This point differs from the work of Berman and Roy.[5] It is also clear that those arguments do not necessarily conclude $\xi\cong-1$ even if an exact SU(3) symmetry is assumed, unlike the work of Brandt and Preparata.[6] What is desired therefore to predict ξ is to obtain the s dependence of the matrix elements of the σ terms. The soft-meson results for these matrix elements are constants of some characteristic masses and we need to know the energy and momentum dependence of the off-mass-shell scattering amplitudes better in detail than those of Weinberg's linear expansion in order to obtain the s dependence of the σ matrix elements from the current algebra. But because of the lack of our ability to obtain such off-mass-shell amplitude from the current algebra, we adopt in the following a phenomenological analysis of the S matrix for the scalar form factor $F(s)$.

The form factor $F(s)$ satisfies the elastic unitarity relation

$$\mathrm{Im}F(s)=\rho(s)F(s)A^*(s), \quad (4)$$

where $A(s)$ is the $I=\frac{1}{2}$, s-wave πK scattering amplitude and $\rho(s)$ is the two-body phase-space factor. The solution of (4) can be put as

$$F(s)=[R(s_0)D(s)]^{-1}R(s)D(s_0)F(s_0), \quad (5)$$

where $D(s)$ is the well-known Omnès function evaluated from the s-wave πK phase shifts and $R(s)$ is a real rational function. $F(s)$ has poles when there are bound states that can contribute to the s-wave $K\pi$ states but if $F(s)$ has zeros they must be contained in experimental results as the form factor is an observable quantity. $R(s)$ contains such poles and zeros. The case of no zeros and no poles in $F(s)$ corresponds to $R(s)=\text{const}$ and this situation can be represented by the familiar conventional form $F(s)=F(s_0)D(s_0)/D(s)$. If a scalar $K\pi$ resonance (κ) exists at $s=s_r\simeq(1.1\text{ Bev})^2$ as some experimental analyses suggest,[14] then $\mathrm{Re}D(s_r)=0$. It can be shown that[15] any reasonable estimates of such $D(s)$ from the Chew-Mandelstam approximation[16] of $A(s)=N(s)/D(s)$ and from the current algebra constraints of the s-wave amplitude at the threshold together with the input $f_+(s)$ obtained from K^* dominance give only a small ξ and can not explain the results of the polarization experiments, $\xi\cong-1$.

However, if there is a zero in $F(s)$, say, at $s=s_1$ below the $K\pi$ threshold, then $R(s)=s_1-s$ in (5) and the situation can be represented by $F(s)=F(s_0)\bar{D}(s_0)/\bar{D}(s)$, where $\bar{D}(s)=D(s)/R(s)=(s_1-s)^{-1}\times D(s)$ and takes in the Chew-Mandelstam approximation the following form:

$$\bar{D}(s)=\bar{D}(s_0)-\frac{s-s_0}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}\frac{\rho(s')ds}{(s_1-s')(s'-s_0)(s'-s)}+\frac{\lambda(s-s_0)}{(s_1-s)(s_1-s_0)}. \quad (6)$$

Again the case of a scalar $K\pi$ resonance corresponds to $\mathrm{Re}\bar{D}(s_r)=0$. Since the phase-space integral varies slowly and is small in magnitude below the threshold, the pole term in (6) is dominant around the physical region of K_{l3} decay, $m_l^2\leq s\leq(m_K-m_\pi)^2$. We can thus approximate $D(s)$ fairly well by

$$\bar{D}(s)/\bar{D}(s_0)\simeq(s_1-s_0)/(s_1-s) \quad (7)$$

or, if a κ exists, by

$$\bar{D}(s)/\bar{D}(s_0) \simeq (s_r-s)(s_1-s_0)(s_1-s)^{-1}(s_r-s_0)^{-1}. \tag{8}$$

Existence of such a form-factor zero, as we stressed above, must be supported by experiments. Indeed, we believe that the fact that $\xi(s) \equiv f_-(s)/f_+(s)$ shows almost no s dependence and is roughly -1 throughout the K_{l3} physical region from the more recent polarization measurements[3] is the indication of a zero in $F(s)$ somewhere between $(m_K-m_\pi)^2$ and $(m_K+m_\pi)^2$ and can be explained by the existence of such zero in $F(s)$. Note from the definition (2) that if $\xi(s) \simeq -1$, then $F(s) \simeq f_+(s)(m_K^2-m_\pi^2-s)$. Note further that the soft-meson theorems are not able to predict such a form of $F(s)$.

Since $F(0)=(m_K^2-m_\pi^2)f_+(0)$ from (2), we obtain from (7) and (8) that

$$(m_K^2-m_\pi^2)f_+(s)+sf_-(s)=(m_K^2-m_\pi^2)f_+(0)(s_1-s)/s_1, \tag{9}$$

$$(m_K^2-m_\pi^2)f_+(s)+sf_-(s)=(m_K^2-m_\pi^2)f_+(0)s_r(s_1-s)s_1^{-1}(s_r-s)^{-1}. \tag{10}$$

With the input $f_+(s)$ obtained from K^* dominance, $f_+(s)=f_+(0)m_{K^*}{}^2(m_{K^*}{}^2-s)^{-1}$ which fits the K_{l3} results excellently,[17] we get from (9)

$$\xi(s)=-\frac{(m_{K^*}{}^2+s_1)(m_K^2-m_\pi^2)}{m_{K^*}{}^2s_1}\left(1-\frac{s}{m_{K^*}{}^2+s_1}\right), \tag{11}$$

and from (10)

$$\xi(s)=-(m_K^2-m_\pi^2)\left[\frac{m_{K^*}{}^2+s_1}{m_{K^*}{}^2s_1}-\frac{1}{s_r-s}-\frac{s(s_r-m_{K^*}{}^2-s_1)}{m_{K^*}{}^2s_1(s_r-s)}\right]. \tag{12}$$

These relations determine the ξ and λ parameters which are defined by the expansion of $\xi(s)$ to the s order,

$$\xi(s) \simeq \xi(1+\lambda s/m_\pi^2). \tag{13}$$

We obtain from (11)

$$\xi=-\frac{(m_{K^*}{}^2+s_1)(m_K^2-m_\pi^2)}{m_{K^*}{}^2s_1}, \quad \lambda=-\frac{m_\pi^2}{m_{K^*}{}^2+s_1}, \tag{14}$$

and from (12)

$$\xi=-(m_K^2-m_\pi^2)\left[\frac{m_{K^*}{}^2+s_1}{m_{K^*}{}^2s_1}-\frac{1}{s_r}\right], \quad \lambda=-\frac{m_\pi^2(s_r-s_1)(s_r-m_{K^*}{}^2)}{s_r[s_r(m_{K^*}{}^2+s_1)-m_{K^*}{}^2s_1]}. \tag{15}$$

In Table I, we give the results of ξ and λ for various positions of the form factor zero between $s=(m_K-m_\pi)^2$ and $(m_K+m_\pi)^2$. Note that all of them are in good agreement with more recent experimental result, $\xi=-1.0\pm0.3$, of Bettels et al.,[3] and with the value $\xi=-0.95\pm0.3$ corresponding to the one-dimensional likelihood ($\lambda=0$) analyses of Cutt et al.[3]

To this end we add a few comments. The possibility of a zero in $F(s)$ has been discussed before by Berman and Roy[5] from the soft-meson theorems. But we have seen above that the soft-meson techniques are inadequate to yield a zero in $F(s)$. However, they predict the so-called Adler zero in the total amplitude and the s-wave projection has also a zero. Indeed, the $I=\frac{1}{2}$ s-wave projection of the $K\pi$ amplitude of Ref. 11 vanishes at $s \simeq 0.88(m_K^2+m_\pi^2)$. Table I includes the case of the form factor zero at this point.

If one adopts the usual convention that[18] the poles and zeros of the amplitude $A(s)$ are included as zeros and poles, respectively, in the denominator function $D(s)$ of $A(s)$ and if $F(s)$ is given by this denominator function via the conventional expression mentioned before, then a zero in the

Table I. Numerical results of ξ and λ for various zero positions of the $I=\frac{1}{2}$ s-wave $K\pi$ amplitude.

s_1 $(\mathrm{GeV})^2$	ξ from (14)	ξ from (15)	λ from (14)	λ from (15)
0.2	-1.42	-1.23	-0.018	-0.005
0.231 $[0.88(m_K^2+m_\pi^2)]$	-1.26	-1.07	-0.018	-0.005
0.244 (m_K^2)	-1.21	-1.02	-0.017	-0.005
0.3	-1.04	-0.853	-0.016	-0.005
0.35	-0.93	-0.743	-0.016	-0.005

amplitude will appear as a zero in the form factor. We think that this is an interesting possibility of the origin of the form-factor zero in view of the fact that a generalized Levinson theorem can be shown to hold for $D(s)$ with such a convention.[18] To see the implication of such a form-factor zero to the off-mass-shell extrapolation, we note that the on-mass-shell quantity $[f_+(s)+f_-(s)]f_+^{-1}(0)$ is small in the region $(m_K - m_\pi^2) \leq s \leq (m_K - m_\pi)^2$. Thus the Callan-Treiman relation[19] which holds for $m_\pi^2 = 0$,

$$f_+(s=m_K^2;\ k^2=0,\ P^2=m_K^2)+f_-(s=m_K^2;\ k^2=0,\ P^2=m_K^2)=f_K/f_\pi, \tag{16}$$

would not extrapolate smoothly back to the on-mass-shell point. We note that such a bad extrapolation would imply existance of arbitrary subtractions in the off-mass-shell form factor

$$i\int d^4x\, e^{ikx}\langle 0|T[\partial_\mu A_\mu^{(\pi)}(x)V_\nu^{(K)}(0)]|K^+(p)\rangle \equiv \frac{f_\pi m_\pi^2 F_\nu(q^2,k^2,p^2=m_K^2)}{\sqrt{2}(m_\pi^2-k^2)}. \tag{17}$$

This can be seen from

$$q_\nu F_\nu(q^2,k^2,p^2=m_K^2)=F(q^2,k^2,p^2=m_K^2)+C\frac{f_K m_K^2}{f_\pi m_\pi^2}(m_\pi^2-k^2), \tag{18}$$

by repeating a similar argument as that Geffen[20] has used for the four-point function. In (18), $F(q^2,k,p^2=m_K^2)$ is the off-mass-shell form factor corresponding to (2) and defined similarly as in (17) and C is a constant depending only on the parameters of the SU(3)⊗ SU(3)-breaking Hamiltonian.

Finally, we remark that our results for ξ and λ are very insensitive to the existence of a κ meson around 1 BeV and to the exact zero position of the amplitude between the branch points $(m_K-m_\pi)^2$ and $(m_K+m_\pi)^2$. It is gratifying that such a simple S-matrix consideration goes in the right direction.

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[1]See for instance W. J. Willis, in *Proceedings of the International Conference on Elementary Particles, Heidelberg, Germany, 1967,* edited by H. Filthuth (North-Holland, Amsterdam, 1968).

[2]T. Eichten *et al.* ($X2$ collaboration), Phys. Lett. 27B, 586 (1968; D. Haidt *et al.* ($X2$ collaboration), Phys. Lett. 29B, 696 (1969).

[3]Earlier summaries for the polarization measurements can be found in Ref. 1 and in J. W. Cronin, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, Austria, September 1968,* edited by J. Prentki and J. Steinberger (CERN Scientific Information Service, Geneva, Switzerland, 1968); and more recent results can be found in J. Bettels *et al.* ($X2$ collaboration), Nuovo Cimento 56A, 1106 (1968); D. Cutts *et al.*, Phys. Rev. 184, 1380 (1969).

[4]Earlier theoretical references on the K_{l3} form factors can be found in K. Kang, Phys. Rev. Lett. 21, 857 (1968); S. Weinberg, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, Austria, September 1968,* edited by J. Prentki and J. Steinberger (CERN Scientific Information Service, Geneva, Switzerland, 1968); C. G. Callan, in *Proceedings of the Topical Conference on Weak Interactions, CERN, Geneva, Switzerland, January 1969* (CERN Scientific Information Service, Geneva, Switzerland, 1969).

[5]S. M. Berman and P. Roy, Phys. Lett. 27B, 88 (1968).

[6]R. A. Brandt and G. Preparata, to be published.

[7]M. Ademollo and G. Gatto, Phys. Rev. Lett. 13, 264 (1964).

[8]S. Weinberg, Phys. Rev. Lett. 17, 616 (1966).

[9]S. Adler, Phys. Rev. 137, B1022 (1965).

[10]H. Yabuki, Phys. Rev. 170, 1410 (1968). Further evidences for zero in the amplitude can be found in the references quoted in this paper.

[11]R. W. Griffith, Phys. Rev. 176, 1705 (1968).

[12]S. L. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 224 (1968); M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2196 (1968).

[13]J. A. Cronin and K. Kang, Phys. Rev. Lett. 23, 1004 (1969).

[14]T. G. Trippe *et al.*, Phys. Lett. 28B, 203 (1968).

[15]K. Kang, I. J. Muzinich, and R. H. Socolow (unpublished).

[16]G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

[17]See Ref. 1 and also C. Rubbia, in *Proceedings of the Topical Conference on Weak Interactions, CERN, Geneva, Switzerland, January 1969* (CERN Scientific

Information Service, Geneva, Switzerland, 1969).
[18]See for instance P. D. B. Collins and E. J. Squires, *Regge Poles in Particle Physics* (Springer, New York, 1968), p. 144.
[19]C. G. Callan and S. B. Treiman, Phys. Rev. Lett. 16, 153 (1966); V. Mathur, S. Okubo, and L. Pandit, Phys. Rev. Lett. 16, 371 (1966).
[20]D. A. Geffen, Phys. Rev. Lett. 23, 897 (1969).