West, Ref. 7.
[10]We can also show that $[d/d(x\cdot P)]f_2(x\cdot P) < 0$ for $x\cdot P > 0$ if $F_2(\omega)$ is a monotonically decreasing function.
[11]H. T. Nieh, unpublished.
[12]G. B. West, Phys. Rev. Lett. 24, 1206 (1970).
[13]C. W. Gardiner and D. P. Majumdar, Phys. Rev. D 2, 151 (1970).
[14]$\int_0^1 d\omega\, F_2(\omega) = \langle Q_p^2\rangle$ in a parton model where $\langle Q_p^2\rangle$ is in the mean squared charge of a parton. Bjorken and Paschos, Ref. 3.
[15]$r_{\text{ch}} = 0.805 \pm 0.011$ F. L. Hand, D. Miller, and R. Wilson, Rev. Mod. Phys. 35, 335 (1963).

RULES FOR CONSTRUCTING DUAL AMPLITUDES*

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The projected propagator of the dual-resonance model is presented. It is used to give an operator proof of duality and to construct various tree and loop operators.

An attractive attitude towards the n-point functions of the dual-resonance model is that they provide the Born terms of a theory of hadrons. Much has already been done to implement this idea. Multiparticle tree graphs have been factorized,[1] thereby yielding the level structure of the model as well as operator expressions for vertices and propagators. All one-loop diagrams have been constructed[2] and renormalized[3,4] and multiloop diagrams have been classified[5] in terms of four primitive loop operators. Despite these achievements, the construction of a complete theory has been impeded by the technical difficulties associated with so-called spurious states.[1,6] Their contributions must be eliminated from the basic operators before a completely dual theory can be formulated. In this paper new forms for the projection operator and the projected propagator are presented. These, together with a vertex operator previously obtained,[7] constitute the complete set of operators required to construct arbitrary dual-resonance diagrams. The projected propagator is used to present an operator proof of duality, to construct multiparticle tree diagrams for arbitrary external states, and to construct simple and useful expressions for the primitive loop operators of the model. These are all the ingredients required to construct any multiloop diagram.[8] Most of the calculational details and some of the basic formulas will be left for a later publication.[9]

The states of the dual-resonance model can be described by vectors in the Hilbert space generated by four-vector creation operators $a_\mu^{\dagger(n)}$, $n = 1, 2, \cdots$. Some of these states are spurious in that they do not couple to any number of the original on-shell external scalar particles. (These scalar particles are described by the ground state of the Hilbert space.) The spurious states are generated by the operator[6]

$$A^\dagger(-p) = L_0(p) - L_+(p) \tag{1}$$

acting on an arbitrary state, where

$$L_0 = R - \tfrac{1}{2}p^2 = \sum_{n=1}^{\infty} n a^{\dagger(n)}\cdot a^{(n)} - \tfrac{1}{2}p^2$$

and

$$L_+(p) = L_-^\dagger(-p) = p\cdot a^{\dagger(1)} + \sum_{n=1}^{\infty} [n(n+1)]^{1/2} a^{\dagger(n+1)}\cdot a^{(n)}.$$

This is due to the fact that $A(p)$ annihilates any vector in the Hilbert space that describes a tree

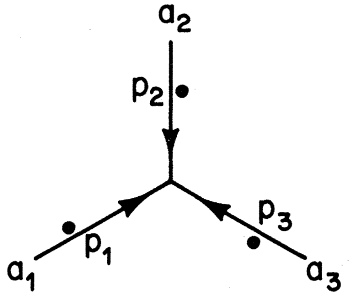

FIG. 1. The symmetric vertex describing the coupling of three arbitrary states.

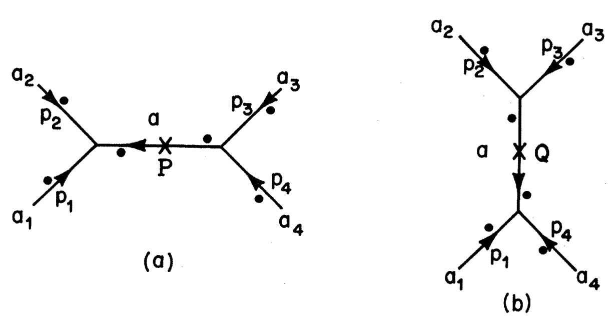

FIG. 2. The four-point operator in two dual configurations.

with external on-shell ground-state scalars. In order to construct completely dual operator rules, not restricted to multiperipheral configurations, it is necessary to eliminate contributions due to the spurious states. To this end, the projection operator onto the space of nonspurious states has been introduced[6, 10]:

$$P(p)=I-[A^\dagger(-p)-\alpha_0]\{A(p)[A^\dagger(-p)-\alpha_0]\}^{-1}A(p). \tag{2}$$

A symmetric vertex operator describing the coupling of three arbitrary states has been obtained by factorization of the tree diagrams.[7] In the configuration of Fig. 1 it is given by

$$V(p_1,p_2,p_3;a_1,a_2,a_3)=g\langle 0|\exp\{(a_1|p_2)+(a_2|p_3)+(a_3|p_1)+(a_1|M_-|a_2)+(a_2|M_-|a_3)+(a_3|M_-|a_1)\}, \tag{3}$$

where we have introduced the matrix notation

$$(a|p)=\sum_{m,n=1}^{\infty}a^{(m)}\cdot\delta_{mn}\frac{p}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{p\cdot a^{(n)}}{\sqrt{n}},\quad (a_i|M_-|a_j)=\sum_{m,n=1}^{\infty}a_i^{(m)}(M_-)_{mn}a_j^{(n)},$$
$$(M_\pm)_{mn}=(mn)^{-1/2}[B(n,\pm m)]^{-1}. \tag{4}$$

This vertex operator may be modified by any "gauge" transformation generated by the operators $A_i(p_i)$. The projected vertex $VP_1^\dagger(-p_1)P_2^\dagger(-p_2)P_3^\dagger(-p_3)$ is unique, however. The dots on the legs of the vertex can be shifted from one side to the other by the twist operator[7,11]

$$\Omega(p)=(-1)^R\exp\{-L_+(p)\}$$

acting on the projected vertex.

Instead of using the projected vertex we find it much more convenient to remove the spurious contributions from the propagator and the external states. The propagator of Fubini and Veneziano,[1]

$$D(p)=\int_0^1 dx\,x^{R-\alpha(p^2)-1}(1-x)^{\alpha_0-1} \tag{5}$$

contains the spurious states. [Our conventions are such that $\alpha(s)=\alpha_0+\frac{1}{2}s$.] For incorporating projections and for constructing cyclic-symmetric amplitudes it is more convenient to work with the twisted propagator $D(p)\Omega(p)$. We now present our main result, the projected twisted propagator:

$$\mathfrak{D}(p)=P^\dagger(-p)D(p)\Omega(p)P(p)=\int_0^1 dx\left(\frac{x}{1-x}\right)^{-\alpha(p^2)-1}(1-x)^{A^\dagger(-p)-1}\left(\frac{x}{x-1}\right)^R(1-x)^{A(p)-1}. \tag{6}$$

This formula can be derived by using the SU(1, 1) algebra[6]

$$[L_0,L_\pm]=\pm L_\pm,\quad [L_+,L_-]=-2L_0, \tag{7}$$

and an integral representation of the hypergeometric function to rewrite Eq. (2) in the form

$$P(p)=\frac{-\pi\alpha_0}{\sin\pi\alpha_0}\int_c\frac{du\,dv}{(2\pi i)^2}(1+u+v)^{-\alpha_0-1}(-u)^{A^\dagger(-p)-1}(-v)^{A(p)+\alpha_0-1}, \tag{8}$$

where the u and v contours each enclose the origin and run to infinity. Once derived, it is easy to verify that the propagator in Eq. (6) has the same matrix elements between external scalar trees as the twisted Fubini-Veneziano propagator. Also, it is gauge invariant since the effect of multiplying $\mathfrak{D}(p)$ on the left by $z^{A(p)-\alpha_0}$ is to effect the transformation $x\to xz/[1-x(1-z)]$ and leave $\mathfrak{D}(p)$ unchanged.

The first application of Eq. (6) to be considered is an operator proof of duality. Duality (for us) means the equality of the operator products depicted in Figs. 2(a) and 2(b) when evaluated between projected states; i.e.,

$$\{V(p_1,p_2,P;a_1,a_2,a)\mathfrak{D}(a,P)V(p_3,p_4,-P;a_3,a_4,a^\dagger)-V(p_2,p_3,-Q;a_2,a_3,a)\mathfrak{D}(a,Q)V(p_4,p_1,Q;a_4,a_1,a^\dagger)\}$$
$$\times P_1^\dagger(-p_1)P_2^\dagger(-p_2)P^\dagger(-p_3)P_3^\dagger(-p_4)=0. \tag{9}$$

Once this identity is established, it follows that any diagram constructed from V and $\mathfrak{D}$ will be dual (i.e., identical to all diagrams related to it by duality transformations) up to a gauge transformation, which is of no consequence when external projected states are attached. The proof of Eq. (9) is obtained by explicitly calculating the four-point operator corresponding to Fig. 2(a) and performing gauge transformations on the external lines so as to yield an operator that is manifestly invariant under the cyclic transformation $(a_i, p_i) \to (a_{i+1}, p_{i+1})$. The possibility of finding such gauge transformations is the essence of the proof. We present here a much more general result—namely, an explicitly cyclic-symmetric N-particle tree diagram for arbitrary external states[12]:

$$F_N(p_i, a_i) = \int dV_N \prod_{1 \le i < j < N} u_{ij}^{-\alpha_{ij}-1} \langle 0| \exp\{\sum_{i=1}^{N} (a_i|\tilde{P}_i) + \sum_{i<j} (a_i|X(i,j)|a_j)\}, \tag{10}$$

where

$$|\tilde{P}_i) = |p_{i+1}) + \sum_{j=1}^{N-3} (u_{i,i+1} u_{i,i+2} \cdots u_{i,i+j}) |p_{i+j+1}),$$

$$X(i,j) = X(j,i) = M_-(1-u_{i,i+1})M_+(1-u_{i+1,i+2})M_+ \cdots M_+(1-u_{j-1,j}),$$

and (u) represents the matrix $(u)_{mn} = u^n \delta_{mn}$; u_{ij} are the standard variables of the N-point function and dV_N is the standard volume element. Thus the nonoperator part of (10) is just the usual N-point function for external scalars. The operator F_N was constructed by evaluating a product of vertices and projected propagators corresponding to a particular tree graph and performing the gauge transformations needed to make it explicitly cyclic symmetric. We repeat that this symmetry in the case $N=4$ constitutes a proof of duality.

The evaluation of tree diagrams for particular external states requires matrix elements of F_N between projected occupation-number states. The special case of on-shell states belonging to the leading Regge trajectory (states generated by the first creation operator $a^{\dagger(1)}$) is particularly simple. In this case the projection is trivial and the amplitude may be obtained by the formal operation

$$\left(\sum_{i=1}^{N} \epsilon_{\mu\nu\cdots\lambda}(p_i) \frac{\partial}{\partial a_{i,\mu}^{(1)}} \frac{\partial}{\partial a_{i,\nu}^{(1)}} \cdots \frac{\partial}{\partial a_{i,\lambda}^{(1)}} \right) F_N(p, a) \Big|_{a_i=0}, \tag{11}$$

where $\epsilon_{\mu\nu\cdots\lambda}(p_i)$ is the polarization tensor for the state of spin J_i on the leading trajectory.

The operators in Eqs. (3) and (6) can also be used to calculate diagrams with loops. As has been shown elsewhere,[5] all diagrams of a dual model can be constructed from four primitive loop operators. For example, all planar diagrams can be obtained by attaching the tadpole operator in Fig. 3 to a tree. This operator is not hard to calculate using Eqs. (3) and (6). The result is

$$T(a) = 4\pi^2 g \int_0^1 \frac{dw}{\ln^2 w} w^{-\alpha_0 - 1} [f(w)]^{-4} \langle 0| \exp\{(a|[1-w]E[1-w]|a)\}, \tag{12}$$

where

$$f(w) = \prod_{n=1}^{\infty} (1-w^n)$$

and E represents the "elliptic" matrix

$$(E)_{mn} = \frac{1}{2(mn)^{1/2} \ln w} + \tfrac{1}{2} \left\{ M_+ \frac{(w)}{1-(w)} M_- + M_-^{\mathrm{T}} \frac{(w)}{1-(w)} M_+ \right\}_{mn} \tag{13}$$

and

$$\left(\frac{(w)}{1-(w)} \right)_{mn} = \frac{w^n}{1-w^n} \delta_{mn}.$$

When attached to a tree of external scalar particles, $T(a)$ gives rise to the known form of the planar one-loop diagram,[4] including the famous factor of $1-w$ discovered by Bardakci, Halpern, and Shapiro.[2]

The matrix E generates the elliptic functions that appear in planar diagrams; in fact, it can also be written in the form

$$(E)_{mn} = \frac{1}{2} \frac{(mn)^{1/2}}{m!\,n!} \left(-\frac{\partial}{\partial x} \right)^m \left(-\frac{\partial}{\partial y} \right)^n \ln \left[\frac{y-x}{\psi(x/y, w)} \right] \Big|_{x=y=1}. \tag{14}$$

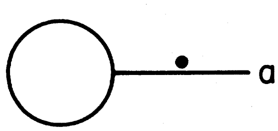

FIG. 3. The tadpole operator $T(a)$.

$\psi(x, w)$ has been previously defined in Ref. 4. One of several equivalent forms for it is

$$\psi(x, w) = -\frac{\ln w}{\pi} \sin\left(\frac{\pi \ln x}{\ln w}\right) \prod_{n=1}^{\infty} \frac{[1-q^{2n} \cos(2\pi \ln x/\ln w) + q^{4n}]}{(1-q^{2n})^2}, \tag{15}$$

where $q = \exp\{2\pi^2/\ln w\}$. The tadpole operator requires renormalization {because of the factor $[f(w)]^{-4}$}, and a counterterm is easily constructed. One simply replaces E by $\tilde{E}$, where $\tilde{E}$ is given by Eq. (14) with $\psi(x)$ replaced by

$$\tilde{\psi}(x, w) = -\frac{\ln w}{\pi} \sin\left(\frac{\pi \ln x}{\ln w}\right), \tag{16}$$

which exponentially approximates $\psi(x)$ near $w = 1$.[3,4]

In conclusion, we believe that the projected propagator in Eq. (6) is a useful discovery, which makes the calculation of any dual-resonance-model diagram possible (but not easy). We have used it, for example, to construct an expression of structure similar to Eq. (12) for the nonplanar self-energy operator. We have also found the nonorientable tadpole operator. These, and other details, will be presented later.[9] Finally, we remark that it is now quite easy to write operator expressions for multiloop diagrams. It is still quite difficult, however, to carry out explicitly the internal operator arithmetic. This would require calculating vacuum matrix elements (or traces) of products of operators some of which are exponentials of quadratic forms in the creation and annihilation operators [as in Eq. (12)].

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[1]S. Fubini and G. Veneziano, Nuovo Cimento 64A, 811 (1969); K. Bardakci and S. Mandelstam, Phys. Rev. 184, 1640 (1969); S. Fubini, D. Gordon, and G. Veneziano, Phys. Lett. 29B, 679 (1969).

[2]K. Bardakci, M. B. Halpern, and J. A. Shapiro, Phys. Rev. 185, 1910 (1969); D. Amati, C. Bouchiat, and J. L. Gervais, Lett. Nuovo Cimento 2, 399 (1969).

[3]A. Neveu and J. Scherk, Phys. Rev. (to be published).

[4]T. H. Burnett, D. J. Gross, A. Neveu, J. Scherk, and J. H. Schwarz, to be published; D. J. Gross, A. Neveu, J. Scherk, and J. H. Schwarz, Phys. Rev. (to be published).

[5]D. J. Gross, A. Neveu, J. Scherk, and J. H. Schwarz, to be published.

[6]C. B. Thorn, to be published; C. B. Chiu, S. Matsuda, and C. Rebbi, Phys. Rev. Lett. 23, 1526 (1969); F. Gliozzi, Lett. Nuovo Cimento 2, 846 (1969).

[7]S. Sciuto, Lett. Nuovo Cimento 2, 411 (1969); L. Caneschi, A. Schwimmer, and G. Veneziano, Phys. Lett. 30B, 351 (1969).

[8]This statement assumes that no unexpected difficulties appear in the calculation of multiloop diagrams.

[9]D. J. Gross and J. H. Schwarz, to be published.

[10]M. Kaku and C. B. Thorn, to be published; R. Brower and J. Weis, Lett. Nuovo Cimento 3, 285 (1970).

[11]D. Amati, M. Le Belloc, and D. Olive, CERN Report No. TH 1118, 1969 (to be published).

[12]This formula has been found by direct factorization of trees for $N = 4$ by G. Carbone and S. Sciuto, Lett. Nuovo Cimento 3, 246 (1970), and in the general case by J. M. Kosterlitz and D. A. Wray, Lett. Nuovo Cimento 3, 491 (1970).