ELASTIC CONSTANTS OF hcp He^4

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Experimental values for the elastic constants of hcp He^4 are derived from longitudinal sound velocities and the compressibility. The derivation makes use of a relation between the elastic constants which is a consequence of the volume independence of the axis ratio of hcp He^4. Good agreement exists between the elastic and calorimetric Debye temperature.

Measurements of the velocity of sound in single crystals of hcp He^4 of known orientation have been reported by several groups.[1-3] The main goal of these experiments is to obtain a complete set of elastic constants for hcp He^4. Once these constants are known one can use this information to obtain quantities like the Debye temperature and the compressibility at absolute zero, Θ_0 and K_0, and the anisotropy of the sound velocity. Theoretical calculations of the sound velocity in various directions have also been published[4,5] and can be used to obtain theoretical estimates for the elastic constants.

In the past, it has been believed that in order to obtain the set of five elastic constants, transverse velocities have to be known, since in particular c_{12} appears only in the expression for the pure transverse mode. We want to show in this Letter that, in fact, the elastic constants of hcp He^4 can be obtained from longitudinal sound data alone, due to a relation that holds for the elastic constants of hcp He^4 [see Eq. (5) below].

The new relation between the elastic constants of hcp He^4 is a consequence of the experimentally established fact that the axis ratio c/a of hcp He^4 stays extremely constant (close to the close-packed value of 1.6330) over a wide range of molar volumes. Various measurements of this ratio were published. Vos et al.[6,7] used the birefringence of hcp He^4 to determine deviations of c/a from the ideal close-packed value. Mills and Schuch[8,9] used x-ray methods and Henshaw,[10] Minkiewicz et al.,[11] and Brun et al.,[12] neutron diffraction to determine to determine the axis ratio. The determinations range from a pressure of 26 to 1050 bar, corresponding to a range in molar volume from 20.9 to 12.5 cm^3/mole. These measurements are summarized in Table I. While there are some discrepancies between the various groups, it is clear that c/a is indeed almost pressure independent. A conservative estimate for the upper limit of the pressure variation is given by

$$|d\ln(c/a)/dP| \lesssim 0.5\times10^{-11}\ \mathrm{cm^2/dyn}. \qquad (1)$$

We now make use of the linear compressibility β, i.e., the relative change in length of a line at an angle γ to the hexagonal axis under hydrostatic pressure. The linear compressibility of a hexagonal crystal is in general anisotropic and of the form (Nye[13])

$$\beta=\beta_1+\beta_2\cos^2\gamma, \qquad (2)$$

where the coefficients β_1, β_2 can be expressed in terms of the elastic constants:

$$\beta_1=(c_{33}-c_{13})/\Delta,$$
$$\beta_2=(c_{11}+c_{12}-c_{33}-c_{13})/\Delta,$$
$$\Delta=c_{33}(c_{11}+c_{12})-2c_{23}^2. \qquad (3)$$

The logarithmic pressure derivative of the axis ratio c/a, for which we use here the notation, $\eta = d\ln(c/a)/dP$, is obviously related to the anisotropy of the linear compressibility and given by

$$\eta=-\beta_2. \qquad (4)$$

For a hexagonal crystal with an axis ratio independent of pressure, we must therefore have the following relation between the elastic constants:

$$c_{11}+c_{12}=c_{33}+c_{13}, \qquad (5)$$

i.e., in this case we have only four independent elastic constants.

Table I. The axis ratio c/a for hcp He^4.

P (bar)	V (cm^3/mole)	c/a	Ref.
26	20.9	1.6320	6
30	20.6	1.6320	7
73	18.8	1.6323	7
142	17.0	1.6324	7
1064	12.5	1.6324	7
30	20.7	1.628	8
131	17.4	1.627	9
67	18.5	1.63	10
25	21.1	1.638	11
230	16.0	1.6288	12

In order to see how well this relation is obeyed for hcp He^4 one can make a comparison with the volume compressibility at 0 K, K_0. In terms of the β_i, and therefore the elastic constants, one has

$$K_0 = 3\beta_1 + \beta_2, \tag{6}$$

and with Eq. (4),

$$\beta_2/\beta_1 = -3\eta/(K_0 + \eta). \tag{7}$$

The volume compressibility at one of our working volumes, 20.32 cm^3/mole, is $K_0 = 0.31 \times 10^{-8}$ cm^2/dyn, so that one obtains from the estimate (1) the upper limit

$$|\beta_2/\beta_1| \lesssim 0.5 \times 10^{-2}. \tag{8}$$

The exact value of β_2/β_1 is not too well known because of the experimental uncertainty in η; it may well be one or two orders of magnitude smaller than the given limit. One finds by computation that the elastic constants calculated by assuming Eq. (5) agree with those obtained by using Eq. (7) within at least 0.5%. Sound-velocity data at present still introduce errors of the order 3%. Once higher precision sound data are available and also better estimates of η, one might have to use Eq. (7). We should also remark that corrections from adiabatic to isothermal elastic constants and corrections for the temperature dependence of the sound velocity are only of the order of 0.2% and can at present be ignored.

One can now, in principle, obtain the elastic constants c_{11}, c_{33}, c_{13}, and c_{44} from longitudinal sound velocities and use Eq. (5) to obtain c_{12}. The present sound data are, however, not of sufficient accuracy to make this the best possible approach. We prefer instead to use the experimentally determined volume compressibility at absolute zero, K_0, in order to arrive at a set of elastic constants. The volume compressibility of a hexagonal crystal is in general given by Eq. (6). If the relation (5) holds, this reduces to

$$K_0 = 3/(c_{33} + 2c_{13}). \tag{9}$$

The elastic constants are then found in this order: c_{11} and c_{33} are obtained from the longitudinal velocity in the basal plane, and parallel to the c axis, respectively. Relation (9) then gives c_{13} by using the experimental values of K_0 of Jarvis, Ramm, and Meyer.[14] With the knowledge of c_{11}, c_{33}, and c_{13} we can obtain c_{44} from the longitudinal sound velocity (we use the value at 45° to the c axis). c_{12} finally comes from the relation (5).

Table II. Elastic constants of hcp He^4. Units: V, cm^3/mole; c_{ij}, 10^8 dyn/cm^2.

V	c_{11}	c_{33}	c_{12}	c_{13}	c_{44}
19.28[a]	7.6	9.8	4.2	1.98	1.96
20.32[a]	5.5	7.1	2.9	1.31	1.40
20.5[b]	4.7	6.3	3.1	1.42	1.13

[a] Based on Ref. 2.
[b] Based on Ref. 3.

We believe that these constants constitute the best presently available experimental information. The estimated error in these constants is 4%.

Elastic constants that were determined in this way are given in Table II and Fig. 1. The data at $V = 20.32$ and 19.28 cm^3/mole are based on our own velocity measurements[2] and those at $V = 20.5$ cm^3/mole are from the velocity data of Greywall and Munarin.[3] Included in Fig. 1 are also the elastic constants of Crepeau, Heybey, and Lee[1] which were obtained exclusively from sound data. As can be seen from the figure, our data extrap-

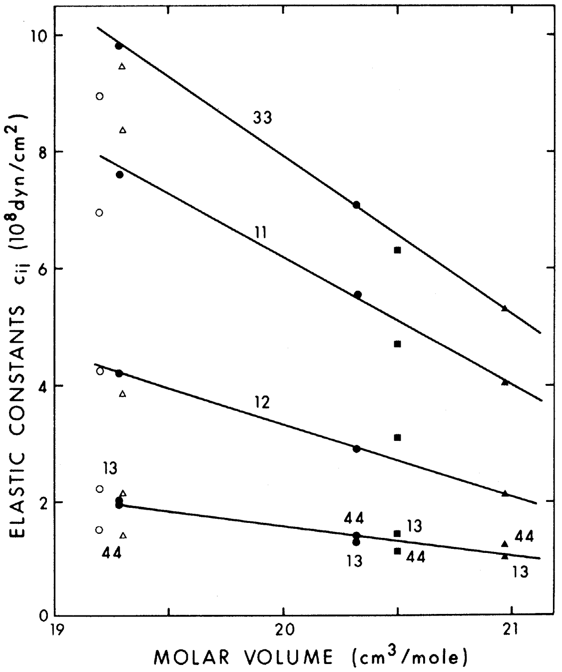

FIG. 1. The elastic constants c_{ij} of hcp He^4 as a function of molar volume. Experimental data: solid triangles, Crepeau, Heybey, and Lee (Ref. 1); solid dots, Wanner and Franck (Ref. 2); solid squares, Greywall and Munarin (Ref. 3). Theoretical calculations: open triangles, Nosanow and Werthamer (Ref. 4); open circles, Gilles, Koehler, and Werthamer (Ref. 5).

olate quite well to the data of Crepeau, Heybey, and Lee. The constants based on Greywall and Munarin's data deviate up to 15% from this agreement, but the general agreement is still fair.

In Fig. 1 we also show the elastic constants based on the theoretical calculations by Nosanow and Werthamer[4] and Gillis, Koehler, and Werthamer. Qualitative agreement with experiment exists; there exist, however, discrepancies in detail. We find, in particular, that relation (5) is fulfilled by the constants of Gillis, Koehler, and Werthamer[5] within the accuracy of their calculation, but not by Nosanow and Werthamer's[4] constants. In the latter case one finds $\beta_2/\beta_1 = 0.085$, which is clearly too large. In none of these calculations was Eq. (5) explicitly used.[15] It would appear that this relation does not necessarily follow if calculations are made only at one density. The question of the validity of Eq. (5) in a particular system will entirely lie with the effective force constants used. In general, the forces will be of such a nature that the axis ratio is not preserved under compression.

An interesting remark can also be made with regard to the Cauchy relations. In a hexagonal crystal, the Cauchy relations are

$$c_{11} = 3c_{12}, \quad c_{13} = c_{44}. \tag{10}$$

These relations hold if the forces are central and if the crystal sites are centers of inversion. The latter condition does not hold in hcp structures; this, however, invalidates only the first but not the second of the Cauchy relations.[16,17] It is now interesting to observe that all experimental data show $c_{13} \approx c_{44}$; the small deviations are well within the error limit. We take this an an indication that the effective forces in hcp He^4 are, at least approximately, central. The first Cauchy relation is not observed, as expected.

We have further used the elastic constants to calculate the Debye temperature at absolute zero, Θ_0, by numerical integration on a computer. In Fig. 2, we compare the "elastic" Debye temperature with the calorimetric determinations of Heltemes and Swenson,[18] Edwards and Pandorf,[19] and Ahlers.[20] We see that agreement exists within the experimental accuracy between the elastic and calorimetric Debye temperatures, for the data based on the results of both Crepeau, Heybey, and Lee,[1] and Wanner and Franck.[2] The data of Greywall and Munarin[3] give an elastic Debye temperature which is 10% low; this probably points to some undetected errors in their data. The identity of the elastic and calorimetric Debye temperature at absolute zero, well established for other materials, therefore holds also for quantum solids.

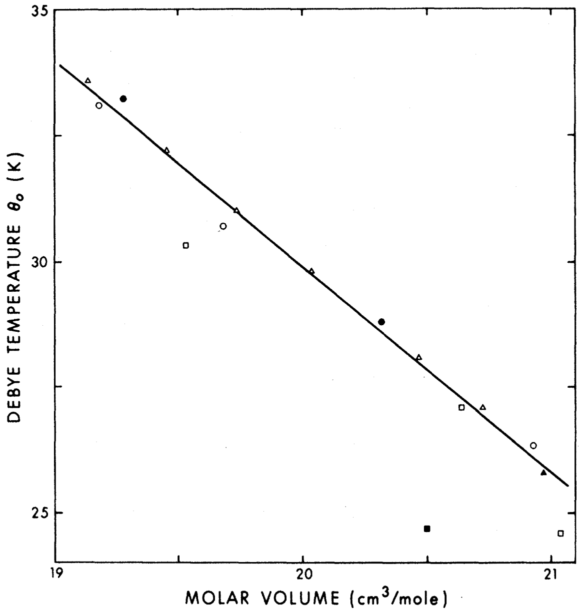

FIG. 2. The Debye temperature at 0 K of hcp He^4 as a function of molar volume. Elastic data: solid triangles, Crepeau, Heybey, and Lee (Ref. 1); solid dots, Wanner and Franck (Ref. 2); solid square, Greywall and Munarin (Ref. 3). Calorimetric data: open squares, Heltemes and Swenson (Ref. 18); open circles, Edwards and Pandorf (Ref. 19); open triangles, Ahlers (Ref. 20).

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ION-WAVE INSTABILITIES AND ANOMALOUS RESISTIVITY*

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A theory of ion-wave instabilities is presented which takes into account the self-consistent inhomogeneities generated by currents flowing along magnetic field lines. It is found that for $T_e \sim T_i$, the minimum unstable current is reduced from the electron to approximately the ion thermal speed. The result is attributed to the drift-wave instability caused by the inhomogeneities produced by the driving current.

For many years it has been known that, for $T_e \sim T_i$, the minimum current for ion-wave instabilities (or the so-called two-stream current instabilities) predicted by the existing theories[1] is too high in comparison with experimental results.[2] In this Letter, we present a theory which takes into account the self-consistent inhomogeneities generated by currents flowing along the magnetic lines of force. In our model, the inhomogeneities are produced by the driving current alone. The plasma is uniform when the driving current vanishes. It should be noted that our model is different from Kadomtsev's[1] model in that the inhomogeneities in his model are independent of the driving current. As has been noted by Bernstein et al.,[1] it is inconsistent to assume currents along the magnetic lines of force and, at the same time, a uniform plasma. The importance of the self-consistent inhomogeneities in a current-carrying plasma is clearly demonstrated by our results. For $T_e \sim T_i$, our theory predicts a minimum unstable current of the order of the ion thermal speed rather than the electron thermal speed predicted by the existing theories.[1] Much of the stellarator data, particularly current work on anomalous resistivity, shows good agreement with our results.[3] It may be noted that the stability criterion predicted by our theory can also be interpreted as the stability criterion for a low-β sheet pinch.

In this communication, we can only outline the general approach and present the main results of our work. Full details will be given in a separate paper. Consider a low-β (β = kinetic pressure/magnetic pressure) collisionless plasma, consisting of electrons and protons, in a constant external magnetic field ($B_0\hat{z}$) with current along the field lines. We consider a one-dimensional model in which all quantities may only depend on the x coordinate. Under three reasonable boundary conditions, it can be shown by extending Harris' solution[4] that for a low-β plasma the most probable self-consistent equilibrium distribution in the presence of a uniform external magnetic field is given by

$$f_{0j} \approx \frac{N_0}{\pi^{3/2} v_{0j}{}^3 \cosh^2(x_0/L)} \left[1 - \frac{2\tanh(x_0/L)}{L}\left(x + \frac{v_y}{\Omega_j} - x_0\right)\right] \exp\{-v_{0j}{}^{-2}[(v_\parallel - V_j)^2 + v^2]\}, \tag{1}$$

where $L = [k(T_e + T_i)/(2\pi N_0 e^2)]^{1/2} c/V$ characterizes the width of the current layer, $V = |V_i - V_e|$ is the uniform relative streaming speed, c the speed of light, $v_{0j} = (2kT_j/m_j)^{1/2}$ the thermal speed, $\Omega_j = q_j B_0/(m_j c)$ the Larmor frequency, and T_j, m_j, and q_j the temperature, the mass, and the charge of jth species, respectively. N_0 is the maximum number density. The boundary conditions are these: (i) The ex-