UNITARY PHENOMENOLOGICAL DESCRIPTION OF THREE-PARTICLE SYSTEMS*

H. Pierre Noyes
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
(Received 14 April 1970; revised manuscript received 25 June 1970)

A general unitary phenomenological description of strongly interacting three-particle systems is developed in terms of two-particle phase shifts and binding energies, two-particle wave functions inside the range of forces, and the three-body wave function in the region where all three force ranges overlap. The two-particle external and internal parameters are unambiguously separated from each other and the same parameters can be determined from many different experiments, while the three-body parameters refer to a specific system at a specific energy.

In order to calculate the wave function or even the on-shell scattering matrix for three-particle states, it is necessary to know not only the binding energies of all two-particle subsystems and their phase shifts at all energies but also the interior wave functions connected to these asymptotic parametrizations.[1] Due to virtual pion emission and absorption, any system containing strongly interacting two-particle subsystems will experience three-particle forces in any region whose perimeter is less than $3\hbar/m_\pi c^2$; complete knowledge of the two-particle subsystems does not determine the dynamics of any strongly interacting three-particle system. It has been shown[3] that if the wave function is known inside this three-body–force region, the exterior three-particle wave function can be calculated by solving convergent one-variable integral equations. In this Letter we convert this demonstration into a practical method for analyzing three-particle systems by proving that it is possible to parametrize this interior wave function in such a way that only three outgoing particles are present (i.e., that the on-shell three-body T matrix obtained by solving this equation is unitary). These interior parameters can be fitted to experiment at a single energy of the three-particle system; they represent an arbitrary (energy-dependent) parametrization of the consequences of unknown two- and three-body forces in the interior region. Additional parameters will have to be included to specify two-particle interior dynamical quantities. These can be independently determined either by making a complete analysis at additional energies of the three-particle system, or by using different third particles as probes. Hence this interior-exterior separation makes possible the construction of the wave functions of two strongly interacting particles inside the range of forces from quantum mechanical observables. To the extent that this nonrelativistic quantum mechanical prescription makes sense, this description should be the same whatever the third particle in the system.

The Wick argument[2] tells us that whenever the outgoing waves in the two other Faddeev channels (which are the physical mechanism that probe the interior two-particle wave functions in the exterior three-particle region)[3,4] contain momenta exceeding pion production threshold within the system probed, this three-particle description breaks down. But even in this situation, the hadron "soup" which is being probed may have average properties independent of the mode of excitation. To the extent that this is true, our analysis will still give correctly the probability of finding that part of the "soup" which is connected to two, and only two, specified hadrons in the asymptotic region. The a priori limit of validity of our description is for distances averaged over regions of order $\hbar/m_\pi c$, or momenta less than $m_\pi c$; only the study of specific systems will reveal whether it holds down to shorter distances and for higher momenta.

We assume[5] that the two-particle half-off-shell t matrices are bounded by

$$t_\alpha{}^l(p,k;p^2)=\tau_\alpha{}^l(p)\int_{\mu^2}^\infty d\beta^2\, C(\beta^2)\frac{\beta^2(\beta^2+4k^2)}{(p^2-k^2)^2+2\beta^2(p^2-k^2)+\beta^4}$$
$$=\tau_\alpha{}^l(p)\left[1+(k^2-p^2)\int_{\mu^2}^\infty d\beta^2\, C(\beta^2)\frac{(2\beta^2+p^2-k^2)}{(p^2-k^2)^2+2\beta^2(p^2+k^2)+\beta^4}\right], \tag{1}$$

where

$$\tau_\alpha{}^l(p^2)=\exp[i\delta_\alpha{}^l(p)]\sin\delta_\alpha{}^l(p)/p \text{ and } \int_{\mu^2}^\infty d\beta^2\, C(\beta^2)=1. \tag{2}$$

Since such two-particle t matrices can be shown[1] to satisfy a Lippmann-Schwinger equation in the three-particle Hilbert space, the unitarity of the T matrix follows algebraically[6] if they are used as driving terms in the Faddeev[7, 8] equations. We can extend[6] this unitarity proof to any convergent system of equations for the on-shell T matrix provided the difference between this system and the Faddeev equations vanishes on shell. We do not have to assume that the three-particle T matrix itself satisfies a Lippmann-Schwinger equation, and therefore can allow unspecified three-body forces to be present.

The amplitudes $M_{\alpha\beta}$ defined by Faddeev[8] can be shown to satisfy two-variable equations[9, 10] by introducing the coordinates $\vec{p}$ for the interacting pair and $\vec{q}$ for the free particle defined by Lovelace[11] (with the on-shell restriction $p^2+q^2=z$) and making the partial-wave decomposition

$$\langle\vec{p}\vec{q}|M_{\alpha\beta}(z)|\vec{p}_0\vec{q}_0\rangle=\sum_{l\lambda}M_{l\lambda}{}^{\alpha\beta}(p,q;z)Y_{Jl\lambda}{}^{M}(\hat{p},\hat{q}). \tag{3}$$

Since the outgoing wave is generated from this amplitude by multiplying by the singular factor $(p^2+q^2-z)^{-1}$, we can define an interior wave function in momentum space by subtracting out the value at the singularity; explicitly,

$$I_{l\lambda}{}^{\alpha\beta}(p,q;z)\equiv M_{l\lambda}{}^{\alpha\beta}(p,q;z)-M_{l\lambda}{}^{\alpha\beta}(p,(z-p^2)^{1/2};z)\equiv(p^2+q^2-z)F_{l\lambda}{}^{\alpha\beta}(p,q;z)T_{l\lambda}{}^{\alpha\beta}(p;z). \tag{4}$$

If $I_{l\lambda}{}^{\alpha\beta}$ is known, the three-particle on-shell T matrices satisfy

$$\begin{aligned}M_{l\lambda}{}^{\alpha\beta}(p,(z-p^2)^{1/2};z)\equiv T_{l\lambda}{}^{\alpha\beta}(p;z)&\\ =t_\alpha{}^{l}(p,p_0;p^2)2(z-p^2)^{-1/2}\delta(p^2+q_0{}^2-z)\delta_{\alpha\beta}\delta_{ll_0}\delta_{\lambda\lambda_0}\delta_{\alpha\alpha_0}&\\ -(z-p^2)^{-1/2}\int_0^\infty dp'^2\int_{q_-{}^2}^{q_+{}^2}dp'^2\,t_\alpha{}^{l}(p,\bar{p};p^2)\sum_{\gamma\neq\alpha}\sum_{l'\lambda'}K_{l\lambda l'\lambda'}{}^{\alpha\gamma}F_{l'\lambda'}{}^{\gamma\beta}(p',q';z)T_{l'\lambda'}{}^{\gamma\beta}(p';z)&\\ -(z-p^2)^{-1/2}\int_0^\infty dp'^2\int_{q_-{}^2}^{q_+{}^2}dq'^2\frac{t_\alpha{}^{l}(p,\bar{p};p^2)}{p'^2+q'^2-z}\sum_{\gamma\neq\alpha}\sum_{l'\lambda'}K_{l\lambda l'\lambda'}{}^{\alpha\gamma}T_{l'\lambda'}{}^{\gamma\beta}(p';z),&\end{aligned} \tag{5}$$

where

$$\bar{p}^2=p'^2+q'^2-z+p^2,\quad q_\pm=p'\tan\mu_{\alpha\gamma}\pm(z-p^2)^{1/2}\sec\mu_{\alpha\gamma},\quad \cos\mu_{\alpha\gamma}=[m_\alpha m_{\gamma}/(m_\alpha+m_\gamma)(m_\gamma+m_{\gamma'})]^{1/2}, \tag{6}$$

and $K_{l\lambda l'\lambda'}{}^{\alpha\gamma}$ are purely geometrical recoupling coefficients.[9]

As already noted, the algebraic form of these equations guarantees that the T matrix determined from them will be unitary provided that the terms by which they differ from the Faddeev equation vanish in the on-shell limit. This happens automatically for those terms which are independent of the interior functions F, since these are proportional to

$$(p^2+q^2-z)^{-1}[t_\alpha{}^{l}(p,\bar{p};z-q^2)-t_\alpha{}^{l}(p,\bar{p};p^2)], \tag{7}$$

and Kowalski[12] has shown that this difference is always proportional to (p^2+q^2-z) times a remainder function which itself vanishes on-shell. Equation (4) has removed the $(p^2+q^2-z)^{-1}$ singularity from the terms proportional to the F's by definition. Therefore any (convergent) complete set of functions which have no discontinuity across any of the three- or two-particle branch cuts in the on-shell limit can be used to expand the interior wave function.

This proof of unitarity still fails if the operator for inverting Eq. (5) does not exist. The existence of this operator was proved in CS, but the proof was cumbersome and required a finite range cutoff R. By using the bound [Eq. (1)] justified above, the kernel in the last term of Eq. (5) which has to be integrated over p' becomes

$$\frac{\tau_\alpha{}^{l}(p)}{(z-p^2)^{1/2}}\left[\int_{L_-}^{L_+}K_{l\lambda l'\lambda'}{}^{\alpha\gamma}\frac{dy}{y}+\int_{\mu^2}^{\infty}d\beta^2 C(\beta^2)\int_{L_-}^{L_+}dy\frac{(2\beta^2-y)}{(y+\beta^2)^2+4\beta^2p^2}K_{l\lambda l'\lambda'}{}^{\alpha\gamma}\right]\equiv Q_{l\lambda l'\lambda'}{}^{\alpha\gamma}(p,p'), \tag{8}$$

where

$$L_\pm=\{[p'\pm(z-p^2)^{1/2}\sin\mu_{\alpha\gamma}]^2-p^2\cos^2\mu_{\alpha\gamma}\}/\cos^2\mu_{\alpha\gamma}. \tag{9}$$

The asymptotic convergence of this kernel follows immediately from the fact that for $z>p^2$ the limits have complex conjugate imaginary parts and hence that the result of the y integration is bounded by $\frac{1}{2}\pi$; by making the change of variable $(p^2-z)^{1/2}=r\sin\varphi$, $p'=r\cos\varphi$, it then follows that if $\tau(p)$ is bounded[5] by A/p^2, the integral $\int^\infty dp^2\int^\infty dp'^2\,Q^2(p,p')$ converges at least as well as $\int^\infty r^3dr/r^6$ at the upper limit. The remaining singularity across the physical three-particle branch cut can be handled in the same way as the singularities in the Faddeev-Lovelace[11] (or Amado[13] or Mitra[14]) equations for separable interactions. Because we have kept the interacting pair in the physical region and allowed the free particle to become virtual, only this singularity occurs; "potential singularities"[15] are absent, and only physically accessible wave functions need be known. The implicit limitation to systems with no two-body bound states is easily removed[16] by separating out the bound-state poles in the two-particle t matrices and coupling in the additional terms they contribute to Eq. (5). The appropriate branch cuts for elastic scattering and rearrangement collisions then appear automatically in the unitarity relation.

Since our proof of convergence given above guarantees the existence of a resolvent kernel for Eq. (5) even in the zero-range limit $\mu^2 \to \infty$ for the two-particle interactions, in this limit Eq. (5) provides one-variable integral equations for the three-particle T matrix using only physical two-particle phase shifts and binding energies formally valid at any energy. Whether this is a good physical approximation will depend on whether the interior effects are physically significant or not[17]; in particular, we know that this limit cannot be used for three-particle bound states since Thomas[18] has shown that the zero-range limit gives infinite binding to the ground states of such systems.

This Letter provides the necessary[4] formalism for the inclusion of whatever knowledge is available about two-particle systems in three-body calculations. Three types of parameters occur. The two-particle scattering amplitude $\tau(p)$ is available from experiment for stable two-particle systems, but must be determined from three-particle experiments in the case of unstable particles. Our formalism allows this to be done uniquely, utilizing all interference terms in all regions of the Dalitz plot, since the remaining parameters can be measured independently. The second set of parameters refers to the interior two-particle wave functions for bound and scattering states. We have demonstrated that, in principle, these can be determined from three-particle states, and their uniqueness verified by showing that the same parameters are determined at different three-particle energies or for different third-particle probes. Because of pion production, these wave functions are, strictly speaking, measured only in the sense of an average over distance of order $\hbar/m_\pi c$, but the description might turn out to have a higher accuracy than that. Finally, there are parameters which refer to the three-particle interior region. These can be computed once some assumption is made about the two- and three-body forces in this region, or assigned phenomenologically to fit three-particle experiments at a single energy. If they are chosen appropriately, strong energy dependence of these parameters would demonstrate genuine three-particle "resonances" not due to the pairwise interactions of the exterior region. Since they refer to a region whose perimeter is approximately $3\hbar/m_\pi c$ in length, a small number of such parameters must suffice at low energy; otherwise it would be possible to measure details of the interior wave function smaller than any exterior wavelength. Study of the exterior region in the three-body problem has demonstrated that we do not yet know whether the basic two-body interactions are local or nonlocal,[4] but now has yielded a method for investigating that question experimentally. It is to be hoped that the study, both theoretical and experimental, of the parameters which describe the interior three-particle region will clarify still further the extent of our understanding of strong interactions.

---

*Work performed under the auspices of the U. S. Atomic Energy Commission.

[1]This information suffices to construct the driving terms for the Faddeev equations, as shown in H. P. Noyes, Phys. Rev. Lett. 15, 538 (1965), and more fully in Progr. Nucl. Phys. 10, 355 (1968).

[2]G. C. Wick, Nature 142, 993 (1938). If the lightest strongly interacting quantum is the pion of mass m_π the generalization of the two-particle force range $r \lesssim c\Delta t \lesssim c\hbar/\Delta E \lesssim \hbar/m_\pi c$ to three-body forces is immediate.

[3]H. P. Noyes, Phys. Rev. Lett. 23, 1201 (1969), hereinafter referred to as CS.

[4]Cf. H. P. Noyes, in *Three Body Problem in Nuclear and Particle Physics,* edited by J. S. C. McKee and P. M. Rolph, (North-Holland, Amsterdam, 1970), pp. 2 and 494, and references therein.

[5]Since the wave functions of the two-particle subsystems inside their range of forces correspond to average values over regions of order $\hbar/m_\pi c$, we allow no singularities in either coordinate or momentum space. Short-range behavior in a local interaction no worse than $1/x$ suffices to give the A/p^2 on-shell bound for $\tau(p)$ used below, while long-range behavior bounded by $e^{-\mu x}$ is sufficient to give the off-shell bound in Eq. (1), but the mathematical details of what class of interactions meet our criteria are of no interest here; in particular, by focusing on the wave function (or half-off-shell t matrix) as basic, we avoid specifying whether the two-particle interaction is local or nonlocal. A specific interaction might require Eq. (1) to be supplemented by additional, less singular, terms before it would become adequate as a *representation* for t, but we believe it suffices as a *bound*. If it should prove impossible to represent experimental data under these restrictions, we would take this as evidence that we

do not know how to construct a meaningful nonrelativistic limit for elementary particle interactions.

[6]These algebraic details will be submitted for publication elsewhere. A still simpler algebraic proof of unitarity for the Faddeev equations has since been called to the attention of the author by K. L. Kowalski; it is contained in D. Z. Freedman, C. Lovelace, and J. M. Namyslowski, Nuovo Cimento 43A, 258 (1966).

[7]L. D. Faddeev, Zh. Eksp. Teor. Fiz. 39, 1459 (1960) [Sov. Phys. JETP 12, 1014 (1961)].

[8]L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory* (Davey, New York, 1965).

[9]A. Ahmadzadeh and J. A. Tjon, Phys. Rev. 139, B1085 (1965).

[10]T. A. Osborn and H. P. Noyes, Phys. Rev. Lett. 17, 215 (1966).

[11]C. Lovelace, in *Strong Interactions and High Energy Physics,* edited by R. G. Moorhouse (Plenum, New York, 1963), p. 437.

[12]K. L. Kowalski, Phys. Rev. Lett. 15, 798, 908(E) (1965).

[13]R. D. Amado, Phys. Rev. 132, 485 (1963).

[14]A. N. Mitra, Nucl. Phys. 32, 529 (1962).

[15]D. D. Brayshaw, Phys. Rev. 176, 1855 (1968).

[16]For details, see Ref. 6.

[17]It is in this limit that our representation for the two-particle interactions is most questionable. It may be that the interior region *always* contributes significant effects, or that the three-particle states may be representable phenomenologically by a more singular set of two-particle interactions than we allow. If so, we insist that the physics lying behind such effects must be studied, and not covered up by using singular phenomenological interactions.

[18]L. H. Thomas, Phys. Rev. 47, 903 (1935).

ERRATA

LIMITS ON MAGNETIC MONOPOLE FLUXES IN THE PRIMARY COSMIC RADIATION FROM INVERSE COMPTON SCATTERING AND MUON-POOR EXTENSIVE AIR SHOWERS. W. Z. Osborne [Phys. Rev. Lett. 24, 1441 (1970)].

Equation (5) should read

$$I_M(E(k)) = (\lambda_M/R)I_\gamma(k).$$

MULTIPION PRODUCTION IN DEUTERON-DEUTERON INTERACTIONS AT 7.9 GeV/c. A. T. Goshaw and M. J. Bazin [Phys. Rev. Lett. 25, 50 (1970)].

The total number of events measured for the experiment was 1000 events (not 10 000 as stated in the text).

In the formula defining the Reggeized pion propagator on page 53, the α_π is the usual Regge exponent, as $(\cosh\xi_\pi/S_0)^{\alpha_\pi}$, and not a multiplicative factor as printed.