with

$$S_{\mu\nu}=p_{3\mu}p_{3\nu}+p_{1\mu}p_{1\nu},\quad A_{\mu\nu}=p_{3\mu}p_{3\nu}-p_{1\mu}p_{1\nu},\quad R_{\mu\nu}=p_{1\mu}p_{3\nu}+p_{3\mu}p_{1\nu},\quad \tilde{S}_{\mu\nu}=p_{6\mu}p_{6\nu}+p_{4\mu}p_{4\nu},$$

$$\tilde{A}_{\mu\nu}=p_{6\mu}p_{6\nu}-p_{4\mu}p_{4\nu},\quad \tilde{R}_{\mu\nu}=p_{6\mu}p_{4\nu}+p_{4\mu}p_{6\nu},\quad \theta=\text{step function.}$$

It is clear that the π and A_1 levels are unchanged while the spin-2 level and all other even states are doubled. When considering the eight-point function one can see from formula (6) that the structure of spin-3 states and higher is affected. It is then clear that the "leg independence" found in Ref. 1 is due to the oversimplified nature of the problem. As usual the daughter structure is more sensitive to the details of the amplitude and cannot be determined at the present stage. We cannot analyze the two-body channel levels (ρ, f) with the six-point function. Any level structure, if it exists, will appear only at the four-to-four transition in the eight-point function.

Conclusions.—We have been able to write a part of the six-point function that correctly describes the π-A_1 trjectory. As a bonus we can discuss the forms of the four- and five-point amplitudes that are obtained contracting lines. The most interesting result is the level structure that obtains in a model where all physical requirements are obeyed. For the first time to our knowledge, doubling of states lying on the leading trajectory is obtained. It is most amusing that the doubling starts at spin 2 as in the case of the A_2, but we have not yet studied this trajectory.

As a consequence of our results we also believe that other calculations,[9] based on scalar isoscalar bosons that bootstrap themselves, might be radically changed when realistic amplitudes are used.

We acknowledge interesting discussions with our colleagues at the Weizmann Institute and with D. Amati.

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LOW-ENERGY EXPANSION FOR ELASTIC THREE-BODY SCATTERING*

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It is shown that the connected scattering amplitude for three free particles may be written (as the three-body energy E goes to zero) in the form

$$AE^{-1}+BE^{-1/2}+C\ln E+O(1)$$

with A, B, and C completely determined by kinematics and the two-body scattering amplitudes at zero energy.

Although many formal and calculational advances have recently been made in the nonrelativistic quantum mechanical three-body problem, several simple and basic questions remain unanswered. In this Letter we examine one of these—the behavior of the three-body elastic amplitudes as E, the center-of-mass energy, tends to zero. We show that under very general assumptions the connected amplitude has the surprisingly simple expansion

$$T_c=AE^{-1}+BE^{-1/2}+C\ln E+O(1), \tag{1}$$

where A, B, and C are expressed entirely in terms of kinematical factors and the two-body s-wave scattering lengths.[1]

We study the amplitude for three free particles of momenta $\vec{k}_1, \vec{k}_2, \vec{k}_3$ scattering to momenta $\vec{k}_1'$, $\vec{k}_2', \vec{k}_3'$. The total energy is

$$E = k_1^2 + k_2^2 + k_3^2 = k_1'^2 + k_2'^2 + k_3'^2,$$

where for simplicity we have taken equal masses and set $\hbar = 2m = 1$. For an on-shell amplitude, $E \to 0$ is most conveniently studied in terms of the variables $\vec{y}_i$ defined by

$$\vec{k}_i = E^{1/2}\vec{y}_i, \quad \vec{k}_i' = E^{1/2}\vec{y}_i'.$$

The conservation of energy and momentum ($\sum_i \vec{y}_i = 0$) determine the physical range of the y's, independent of E, and hence the limit $E \to 0$ can be studied for fixed y's.

The terms in T_c that diverge as $E \to 0$ come entirely from the first three terms in the connected multiple-scattering expansion. Typical examples of each order are represented in Figs. 1(a), 1(b), and 1(c). The contribution of Fig. 1(a), a double scattering, expressed in terms of the y's is

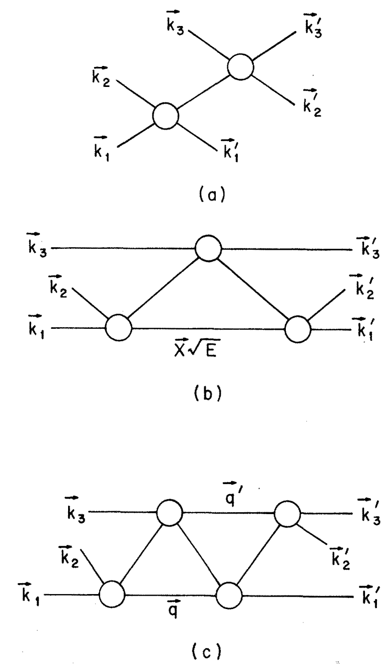

FIG. 1. (a) Typical double, (b) triple, and (c) fourth-order rescattering graphs. The circles indicate two-body t matrices. The momentum labels appropriate to unlabeled lines can be obtained by using momentum conservation and the fact that we work in the center of mass.

$$\frac{\langle \frac{1}{2}E^{1/2}(\vec{y}_1 - \vec{y}_2) | t_3(E(1-\frac{3}{2}y_3^2)) | E^{1/2}(\vec{y}_1' + \frac{1}{2}\vec{y}_3)\rangle}{E[1 - y_1'^2 - y_3^2 - (\vec{y}_1' + \vec{y}_3)^2]} \langle E^{1/2}(\vec{y}_3 + \frac{1}{2}\vec{y}_1') | t_1(E(1-\frac{3}{2}y_2'^2)) | \frac{1}{2}E^{1/2}(\vec{y}_3' - \vec{y}_2')\rangle, \tag{2}$$

where, for example, $\langle \vec{k} | t_1(\epsilon) | \vec{k}'\rangle$ is the off-shell two-body scattering amplitude for the scattering of pair 23 from relative momentum $\vec{k}$ to relative momentum $\vec{k}'$ at energy ϵ. We notice that when these two-body amplitudes have a well-defined nonzero limit as $E \to 0$ the contribution for Eq. (2) goes like $1/E$. To be more precise about the limit $E = 0$, we now state our assumption on the limiting behavior of the two-body amplitudes:

$$\langle \vec{k} | t_j(E) | \vec{k}'\rangle \to \tau_j(E) + O(k^2, k'^2, \vec{k}\cdot\vec{k}')$$

as both $|\vec{k}|$ and $|\vec{k}'| \to 0$, where $\tau_j(E)$ is the s-wave scattering amplitude for the jth pair;

$$\tau_j(E) = 4\pi a_j + i4\pi E^{1/2} a_j^2 + O(E) \text{ as } E \to 0,$$

where a_j is the s-wave scattering length; and finally to ensure convergence of the integrals, we need the weak bound (true except at bound-state poles)

$$\langle \vec{k} | t_j(E) | \vec{k}'\rangle \le C | 1 + |\vec{k} - \vec{k}'| |^{-3/2}.$$

These assumptions are very weak. They are certainly satisfied by scattering amplitudes coming from short-range, nonsingular, central potentials,[2] and are weaker than the usual two-body analyticity assumptions. Using these conditions, Eq. (2) becomes

$$\frac{(4\pi)^2 a_3 a_1}{E[1 - y_1'^2 - y_3^2 - (\vec{y}_1' + \vec{y}_3)^2]} + \frac{i(4\pi)^2 a_3 a_1 [a_3(1-\frac{3}{2}y_3^2)^{1/2} + a_1(1-\frac{3}{2}y_1'^2)^{1/2}]}{E^{1/2}[1 - y_2'^2 - y_3^2 - (\vec{y}_1' + \vec{y}_3)^2]} + O(1) \tag{3}$$

as $E \to 0$. We wish to emphasize that with our choice of variables, the location of the well-known rescattering singularity depends only on the y's.[3] It occurs for Fig 1(a) when $1 - y_1'^2 - y_3^2 - (\vec{y}_1' + \vec{y}_3)^2 = 0$. Hence, the rescattering singularity and the zero-energy singularity are completely independent.

The A of Eq. (1) is simply the coefficient of the E^{-1} term in Eq. (3) summed over the six permutations of 123. The B term of Eq. (1) receives a contribution from all permutations of the second term in Eq. (3) and also from the third multiple-scattering terms. A typical such term, shown in Fig. 1(b), contributes to the connected amplitude the quantity

$$\frac{1}{E^{1/2}}\frac{1}{(2\pi)^3}\int\frac{dx\, t_3t_1t_3}{[1-y_3{}^2-(\vec{y}_3+\vec{x})^2-x^2][1-y_3{}'^2-(\vec{y}_3{}'+\vec{x})^2-x^2]}, \tag{4}$$

where we have defined the loop momentum as $E^{1/2}\vec{x}$, and we have suppressed the momentum and energy dependence of the two-body t matrices. Our assumptions about the two-body t matrix allow us to obtain the divergent part of Eq. (4) as $E\to 0$ by putting $E=0$ under the integral sign. We then obtain for Eq. (4)

$$\frac{a_3a_1a_3}{E^{1/2}}8\int\frac{d\vec{x}}{[1-y_3{}^2-x^2-(\vec{y}_3+\vec{x})^2][1-y_3{}'^2-x^2-(\vec{y}_3{}'+\vec{x})^2]}+O(1). \tag{5}$$

Note the integral will be singular if the rescattering condition is satisfied by the y's. The complete contribution to B of Eq. (1) from the third-order multiple scatterings is seen to be

$$\sum_{\substack{i\neq j\\ k\neq j}} a_ia_ja_kI_{ijk},$$

where I is a kinematical integral of the type given in Eq. (5).

The logarithmic term in Eq. (1) comes from the fourth-order multiple-scattering terms, a typical one of which is shown in Fig. 1(c). It gives

$$\frac{1}{(2\pi)^6}\int\frac{d\vec{q}d\vec{q}'\, t_3t_1t_3t_2}{[E-k_3{}^2-q^2-(\vec{k}_3+\vec{q})^2][E-q^2-q'^2-(\vec{q}+\vec{q}')^2]}\frac{1}{[E-q'^2-k_1{}'^2-(\vec{k}_1{}'+\vec{q}')^2]} \tag{6}$$

The logarithmic singularity is not apparent if we scale the loop momenta by $E^{1/2}$; rather we see that the integral in Eq. (6) is logarithmically divergent at $E=0$, and the divergence comes from the lower limit of the $\vec{q}$, $\vec{q}'$ integration. To obtain the $\ln E$ singularity and its coefficient, one can split the qq' integrations into a part in which q and q' are kept less than some finite cutoff, and the rest which is finite as $E\to 0$. The cutoff integral has a $\ln E$ singularity and can be written

$$a_3a_1a_3a_1\ln E I(y_j, y_j') + O(1) \tag{7}$$

as $E\to 0$, where I is a kinematic expression independent of the particular cutoff. The total contribution to C is obtained from summing over all fourth-order contributions.

It is easily seen, essentially by dimensional arguments, that all higher multiple-scattering terms are finite as $E\to 0$. Further, by analyzing the Faddeev equations, it can be shown that the sum of these terms is finite at $E=0$ even if the multiple-scattering series does not converge, so long as there is no three-body bound state at $E=0$.[4] The unlikely existence of a three-body bound state at $E=0$ simply adds to A the residue of the bound-state pole. The existence of three-body forces in no way changes our results so long as they are of finite range and nonsingular.

If the particles do not all have the same mass, our results remain essentially unchanged. The general form of Eq. (1) is maintained, but the kinematical factors in A, B, C will change. This is to be contrasted with the rescattering singularities which depend strongly on the mass ratio.[3]

Finally we note that an analogous analysis of the N-body connected elastic amplitude may be carried out. In particular the leading divergence as the total center-of-mass energy E vanishes goes like E^{2-N}.

We have carried out our analysis entirely in the framework of nonrelativistic quantum mechanics, but insofar as any relativistic theory possesses a reasonable nonrelativistic limit as $E\to 0$, our results will apply to it.

The low-energy expansion of the three-body elastic amplitude given here has a number of applications. For example, we have used it to understand the analytic properties of three-body decay amplitudes where it leads to some rather surprising threshold behavior.[5] Its usefulness as a low-energy approximation in statistical mechanics is under study.

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*Work supported in part by the U. S. Atomic Energy Commission and the National Science Foundation.

[1]As usual we define $O(g)$ as E tends to zero to mean

that if $f=O(g)$,

$$\lim_{E\to 0}(f/g)$$

is finite.

[2]Cf. M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), p. 285 ff; L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory* (Daniel Davely, New York, 1965), p. 16 ff.

[3]Cf. M. H. Rubin R. Sugar, and G. Tiktopoulos, Phys. Rev. 146, 1130 (1966).

[4]M. H. Rubin, R. Sugar, and G. Tiktopoulos, Phys. Rev. 159, 1438 (1967). In this paper the slightly exaggerated claim is made that the full amplitude is finite at $E=0$, but in fact the proof is only given for the sum of multiple-scattering terms above third order.

[5]R. D. Amado, D. F. Freeman, and M. H. Rubin, to be published.

ERRATA

SUM RULES, STRUCTURE FACTORS, AND PHONON DISPERSION IN LIQUID He^4 AT LONG WAVELENGTHS AND LOW TEMPERATURES. David Pines and Chia-Wei Woo [Phys. Rev. Lett. 24, 1044 (1970)].

A factor 2 is missing from Eqs. (12), (17), and (19). Wherever ξ appears in these equations it should read 2ξ. There is no change in any of the results reported.

MAGNON-PAIR MODES IN TWO DIMENSIONS. P. A. Fleury and H. J. Guggenheim [Phys. Rev. Lett. 24, 1346 (1970)].

On the last line of p. 1347 the words "zeroth moment" should be replaced by "first moment."

Reference 2 should read: "J. B. Parkinson, J. Phys. C: Proc. Phys. Soc., London 2, 2012 (1969). The low-temperature magnon-pair line shape in K_2NiF_4 was calculated and observed independently by S. R. Chinn [thesis, Massachusetts Institute of Technology, 1970, (unpublished)]. The results appear in S. R. Chinn, H. J. Zeiger, and J. R. O'Connor, to be published."

MAGNETIC MOMENT OF A PARTICLE WITH ARBITRARY SPIN. C. R. Hagen and W. J. Hurley [Phys. Rev. Lett. 24, 1381 (1970)].

The following corrections should be noted:

In the definition of $\Delta^{1/2}(\vec{v},R)$ the quantity $\frac{1}{2}\vec{\sigma}\cdot\vec{v}$ should be preceded by a minus sign.

The expression for D which follows Eq. (2) should read $D=\beta(\gamma p+m)$.

The Lagrangian immediately preceding Eq. (5) has three incorrect indices. The corrections are readily identified by taking the $\lambda=1$ limit of Eq. (8).

In the equation following Eq. (5) the first subscript on $(\vec{\sigma}\cdot\vec{\Pi}\vec{\sigma}\cdot\vec{\Pi})$ should be a_i.

In Eq. (6) replace ψ^k by its negative and the last superscript by r_2'. Corresponding to this, all signs in the equation which follows (7) are to be taken positive.

In the final equation the first subscript on the last φ should be a_1.

SPIN FLUCTUATIONS IN NEARLY ANTIFERROMAGNETIC METALS. Tôru Moriya [Phys. Rev. Lett. 24, 1433 (1970)].

The last equation on p. 1435 should read

$$\chi_0''(Q+q,\omega)\simeq(S/8\pi^2)|\nabla(E_{k+Q}-E_k)|^{-1}(\tfrac{1}{2})\{[1+\mathrm{sgn}(\vec{q}\cdot\nabla E_{k+Q})]\Theta(\omega-\vec{q}\cdot\nabla E_{k+Q}) + [1+\mathrm{sgn}(\vec{q}\cdot\nabla E_k)]\Theta(\omega-\vec{q}\cdot\nabla E_k)\}.$$

The name of the author of Ref. 9 should read W. M. Lomer.

S. Doniach [J. Appl. Phys. 39, 483 (1968)] had previously discussed a similar problem by using a special model due to Fedders and Martin. However, the conjecture stated there as to the divergence of the coefficient of linear specific heat at the critical boundary does not seem to be warranted.