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[1]See, for example, D. J. Crennell *et al.*, Phys. Rev. Lett. 19, 1212 (1967); R. Ehrlich *et al.*, *ibid.* 21, 1839 (1968); K. F. Galloway *et al.*, Phys. Lett. 27B, 250 (1969); V. Alles-Borelli *et al.*, Nuovo Cimento 47A, 232 (1967).

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[5]The ($\pi^+ n$ + neutrals) spectrum consists of events having a mass of n + neutral(s) > 1.090 GeV. The (p + neutrals) spectrum contains events with mass of neutrals ≥ 0.5 GeV.

[6]The $n\pi^+$ events are selected with $0.790 \leq \mathrm{MM} \leq 1.090$ GeV while $p\pi^0$ events are required to have $-0.3 \leq \mathrm{MM} \leq 0.4$ GeV, χ^2 probability greater than 10%, and no fit to the elastic hypothesis.

CONSTRUCTION OF PHYSICAL DUAL-RESONANCE MODELS*

V. Rittenberg and H. R. Rubinstein

Department of Nuclear Physics, Weizmann Institute of Science, Rehovot, Israel

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We construct and study six- and eight-charged-pion amplitudes. Factorization properties of the parent π trajectory show the π and A_1 to be simple and the $\pi(2^-)$ to be a double state reminiscent of the A_2 doubling.

The structure of the N-point dual amplitudes is currently the subject of much study. Fubini and Veneziano[1] and others[2] have shown that it is likely that there is a large degeneracy of states in order to maintain duality. One major drawback of the model amplitudes considered is that they lack several basic ingredients that physical amplitudes must possess. Hence, the investigations are performed with the hope that the properties established on the basis of these models will survive the modifications required by realistic amplitudes. As shown below, they do not, and radical changes occur when the intercepts are allowed to be positive. As a consequence, the particle spectrum is changed and in the case under study the degeneracy of the leading trajectory is affected, providing for doubling of the spin-2 and higher states, in a way reminiscent of the A_2 splitting.

First we report on a general method to construct N-point functions for arbitrary trajectories. For reasons stated below we will specialize in the six- and eight-point functions. We have not solved all the standing problems of the N-point functions but we have been able to make a choice that bypasses all difficulties.[3] Our amplitude is an exact representation of the $(\frac{1}{2}N)\pi^+$, $(\frac{1}{2}N)\pi^-$ amplitude when the three-body channels are dominated by the pion trajectory. Our isospin and parity are well defined in all channels and, as long as the contributions of other trajectories are additive (as they are generally believed to be), our solution is the most general one.

(a) All the relevant singularities, (b) Regge behavior, (c) bootstrap condition, and (d) absence of ghosts, are conditions explicitly obeyed by our formulas. We have no analytic expression but a construction procedure only. We have studied in detail the six-point function and some aspects of the eight-point one. One of the most interesting questions is the level structure. Both π and A_1 are simple states. However, the 2^- on the leading trajectory is doubled. These conclusions cannot be affected by the higher-order functions. We have not solved the factorization problem in general. The number of states depends on the number of external legs as opposed to the case in Ref. 1. Nevertheless, for fixed J, there is a finite number of diagrams that contribute to the level structure. Since we have computed only the six-point function we completely determined the degeneracy of the three lowest ones on the leading trajectory. No statement can be made on the ρ and f_0 since one side is always a two-body state and levels cannot be made explicit. One must solve the factorization problem of the eight-point function. We hope to solve this problem. It is most tempting to speculate that positive G-parity states have a different behavior than negative ones. Anyhow, barring unforeseen cancelations it is a prediction of the dual-resonance models that the recurrence of the pion is doubled. Cancelations are not expected since the intercept and analytic structure of other contributions are different.

As byproducts of the calculation, one can obtain the general expression for couplings up to the eight-point function and the general form for the scattering amplitudes with external daughter states. These forms agree with the forms already conjectured by Rubinstein, Squires, and Chaichian.[4] It should be emphasized that our results are based on the simplest and most predictive form of the amplitude. By adding terms that lack lowest-lying poles in some or all channels we can change the multiplicity of some states in both directions. Nevertheless, as in π-π scattering we find it appealing that the simplest minimal form has the aforementioned properties.

Construction of the amplitudes and coupling and pole properties. —We generalize Lovelace's formula for π-π scattering which may be written as follows:

$$A_{1,0}{}^{0,1} = c\,\alpha_{12} B(X_{12}, 1+X_{23}),$$

where

$$X_{ij} = -\alpha_{ij}. \tag{1}$$

In this notation the upper indices represent the units which are added to the arguments of the B function and the lower ones refer to the power of the corresponding arguments which are taken outside of the B function. To obtain the full amplitude one must add all terms corresponding to cyclical and noncyclical permutations of the external lines compatible with the absence of exotic states. Here however, since $A_{1,0}{}^{0,1} = A_{0,1}{}^{1,0}$, the terms obtained are identical. The natural generalization of (1) is

$$A^{m_{12}, m_{13} \cdots m_{N-2,N-1}}_{n_{12}, n_{13} \cdots n_{N-2,N-1}} = \alpha_{12}{}^{n_{12}} \alpha_{13}{}^{n_{13}} \cdots \alpha_{N-2,N-1}{}^{n_{N-2,N-1}} B(\cdots, X_{ij}+m_{ij}, \cdots), \tag{2}$$

where B is the usual N-point function.[3] The order of the arguments is the following:

$$X_{12}, \cdots X_{1,N-2};\ X_{23} \cdots X_{2,N-1};\ X_{N-2,N-1}.$$

The correlation between the exponents n_{ij} and the arguments $X_{ij}+m_{ij}$ is established by the requirement of multi-Regge behavior and absence of ghosts. It can then be proven that a factor α_{pq} implies the addition of one unit to the following arguments X_{ij} of the N-point B function:

$$X_{p+1,q+1}, X_{p+1,q+2}, \cdots, X_{p+1,p-2}; X_{p+2,q+1}, \cdots, X_{p+2,p-2}; X_{q,q+1}, \cdots, X_{q,p-2}$$

(by convention the index $-|m|$ is equivalent to $N-|m|$). Moreover the effect of two external factors $\alpha_{pq}\alpha_{p'q'}$ is additive unless we have $p=p'$ or $q=q'$ (this case will not concern us here). The choice of factors n_{ij} is made so as to ensure that conditions a, b, c, and d states above are fulfilled.

For the six-pion case it can be shown that the unique solution to the problem which is fully Regge behaved term by term in all two- and three-body channels is

$$c_1 \sum \alpha_{12}\alpha_{45} B(X_{12}, X_{13}, X_{14}+1, X_{23}+1, X_{24}+2, X_{25}+1, X_{34}+1, X_{35}, X_{45}) = c_1 \sum A^{0,0,1,1,2,1,1,0,0}_{1,0,0,0,0,0,0,0,1}, \tag{3}$$

where the sum is over all permutations compatible with the duality rules. Notice that the terms in (3) do not have cyclical permutation symmetry. If we relax the condition that every term is fully Regge behaved in all channels, then the "minimal" solution to the problem is given by taking the set of smallest m_{ij} numbers compatible with the aforementioned conditions a, b, c, and d. These solutions are fully Regge behaved but only after cyclical permutation and not term by term. They are of the form

$$\sum \{ c_2 A^{0,1,1,2,1,1,0,0,1}_{1,0,0,0,0,0,1,0,0} + c_3 A^{0,1,0,2,1,2,0,1,2}_{1,0,1,0,0,0,1,0,0} \}. \tag{4}$$

Both forms (3) and (4) reduce to the form (1) when three pions are on a pion pole. For $\pi\pi$-πA_1 scattering the general expression is[5]

$$\{\lambda_1[P_3(1-\alpha_{23}) + P_1(1-\alpha_{12})] + \lambda_1{}^1 P_2\} \epsilon^{A_1} B(X_{12}+1, X_{23}+1). \tag{5}$$

The expression (3) predicts $\lambda_1 = 0$. Because of its simplicity we will adopt it, but there is no deep reason to prefer it. Formula (4) predicts a doubling of the A_1 unless a specific choice is made of c_2 and c_3 in which case factorization can be achieved with one level. Hence, the ratio of the two couplings of the A_1 is free [taking linear combinations of (3) and (4)] in the dual models unless some assumption about the level structure is made.

We wish now to check if there is a generalization of (3) to the eight-point level so that a simple form like (3) for the eight would reduce according to the bootstrap rule to (3). It can be proven that there is no solution like (3) that is fully Regge behaved term by term by term, but there is a "minimal" solution which is

$$\sum\{B^{0,0,1,0,1,1,2,1,2,1,1,0,1,0,0,1,1,2,1,0}_{1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,1}+B^{1,0,0,0,1,0,0,1,2,1,1,2,3,2,1,2,1,1,0,0}_{0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1} -B^{0,1,1,1,1,2,1,2,1,1,0,1,1,0,2,1,1,0,0,1}_{1,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,1,0,0}\}. \tag{6}$$

The same procedure can also be used for the ten-point function, etc. We should like to stress that we have been unable to find a closed expression for the N-point function. Their properties are studied by comparing the six to the eight, eight to ten, etc. Comparing (1), (3), and (6) it is clear that the higher the number of external lines the higher the degree of the external polynomial and the maximum value of m_{ij} needed. This forces a change in the level structure as a function of the number of external legs. We leave this problem for the next section.

An interesting application is to deduce the form of the amplitudes with a smaller number of legs when one external state is a daughter state. Our form for the $\pi\pi\pi\pi\sigma$ scattering amplitude as derived from (3) is

$$\sum[-\tfrac{3}{4}B^{1,0,0,0,1}_{0,0,1,0,0}+B^{1,1,0,0,1}_{0,0,1,0,0}+B^{1,0,0,1,1}_{0,0,1,0,0}+(\tfrac{1}{4}+\tfrac{1}{2}\alpha_{123})B^{0,0,1,2,1}_{1,0,0,0,0}+(\tfrac{1}{4}+\tfrac{1}{2}\alpha_{123})B^{1,2,1,0,0}_{0,0,0,0,1}], \tag{7}$$

where the σ is associated with p_5. The sum is only over noncyclical permutations. It is important to notice that besides the leading term [which is the first term in (7)] we also get nonleading contributions even if we start with a six-pion function which has only leading terms. This is in agreement with the conjecture of Ref. 4 where it was shown that the scattering of external daughters necessitates nonleading terms. It seems then inconsistent to construct formulas for $\pi\pi\pi\pi\sigma$ without them.[6] Of course we cannot exclude the possibility that one σ has a simple form and the other members of the (necessarily) degenerate set contribute all the other pieces, but the total amplitude over the levels must have the form (7).

Factorization properties of our six-point amplitude. – We now analyze the factorization properties of solution (3) using the methods of Ref. 1. Using a more general solution could not diminish the degeneracy unless the most general (but not predictive) forms are used.[7,8] In this case the degeneracy of the leading trajectory can be supressed. In the notation of Ref. 1 our formula 3 reads for the three-body trajectory

$$\int dx dy dz\left\{\frac{\alpha_{12}}{x}\frac{\alpha_{45}}{1-y}\exp\left(-\sum_n\frac{z^n}{n}[P^n Q^n+x^n(1-y^n)-\tfrac{1}{2}]\right)+\frac{\alpha_{23}}{1-x}\frac{\alpha_{56}}{y}\exp\left(-\sum_n\frac{z^n}{n}[P^n Q^n+y^n(1-x^n)-\tfrac{1}{2}]\right)\right.$$
$$\left.+z^2\alpha_{34}\alpha_{25}\exp\left(-\sum_n\frac{z^n}{n}[P^n Q^n-x^n y^n-\tfrac{3}{2}]\right)\right\}x^{-\alpha_{12}}(1-x)^{-\alpha_{23}}y^{-\alpha_{56}}(1-y)^{-\alpha_{45}}z^{-\alpha_{13}-1},$$

where

$$p^n=\sqrt{2}(P_3+P_2x^n),\quad Q^n=\sqrt{2}(P_4+P_5y^n).$$

We restrict ourselves to the leading trajectory. By performing all permutations and neglecting terms proportional to z that do not contribute to the leading trajectory our formula transforms into

$$4\int dx\,dy\,dz\left\{[1+(-)^n]R\tilde{R}\theta(n-2)+\frac{2}{\eta(n-1)}T^n\tilde{T}^n\right\}\frac{(P^1\cdot Q^1)^{n-2}}{(n-2)!}(-1)^n x^{-\alpha_{12}}(1-x)^{-\alpha_{23}}y^{-\alpha_{56}}(1-y)^{-\alpha_{45}}z^{-\alpha_{13}-1+n},$$

where

$$T=\frac{1}{2}\left[\frac{\alpha_{12}}{x}+(-)^n\frac{\alpha_{23}}{1-x}\right][S+A(1-2x)-2(S+R)x(1-x)],\quad \tilde{T}=\frac{1}{2}\left[\frac{\alpha_{45}}{1-y}+(-)^n\frac{\alpha_{56}}{y}\right][\tilde{S}+\tilde{A}(1-2y)-2(\tilde{R}+\tilde{S})y(1-y)],$$

with

$$S_{\mu\nu}=p_{3\mu}p_{3\nu}+p_{1\mu}p_{1\nu},\quad A_{\mu\nu}=p_{3\mu}p_{3\nu}-p_{1\mu}p_{1\nu},\quad R_{\mu\nu}=p_{1\mu}p_{3\nu}+p_{3\mu}p_{1\nu},\quad \tilde{S}_{\mu\nu}=p_{6\mu}p_{6\nu}+p_{4\mu}p_{4\nu},$$
$$\tilde{A}_{\mu\nu}=p_{6\mu}p_{6\nu}-p_{4\mu}p_{4\nu},\quad \tilde{R}_{\mu\nu}=p_{6\mu}p_{4\nu}+p_{4\mu}p_{6\nu},\quad \theta=\text{step function}.$$

It is clear that the π and A_1 levels are unchanged while the spin-2 level and all other even states are doubled. When considering the eight-point function one can see from formula (6) that the structure of spin-3 states and higher is affected. It is then clear that the "leg independence" found in Ref. 1 is due to the oversimplified nature of the problem. As usual the daughter structure is more sensitive to the details of the amplitude and cannot be determined at the present stage. We cannot analyze the two-body channel levels (ρ, f) with the six-point function. Any level structure, if it exists, will appear only at the four-to-four transition in the eight-point function.

Conclusions.—We have been able to write a part of the six-point function that correctly describes the π-A_1 trjectory. As a bonus we can discuss the forms of the four- and five-point amplitudes that are obtained contracting lines. The most interesting result is the level structure that obtains in a model where all physical requirements are obeyed. For the first time to our knowledge, doubling of states lying on the leading trajectory is obtained. It is most amusing that the doubling starts at spin 2 as in the case of the A_2, but we have not yet studied this trajectory.

As a consequence of our results we also believe that other calculations,[9] based on scalar isoscalar bosons that bootstrap themselves, might be radically changed when realistic amplitudes are used.

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LOW-ENERGY EXPANSION FOR ELASTIC THREE-BODY SCATTERING*

R. D. Amado and Morton H. Rubin
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
(Received 27 May 1970)

It is shown that the connected scattering amplitude for three free particles may be written (as the three-body energy E goes to zero) in the form

$$AE^{-1}+BE^{-1/2}+C\ln E+O(1)$$

with A, B, and C completely determined by kinematics and the two-body scattering amplitudes at zero energy.

Although many formal and calculational advances have recently been made in the nonrelativistic quantum mechanical three-body problem, several simple and basic questions remain unanswered. In this Letter we examine one of these—the behavior of the three-body elastic amplitudes as E, the center-of-mass energy, tends to zero. We show that under very general assumptions the connected amplitude has the surprisingly simple expansion

$$T_c=AE^{-1}+BE^{-1/2}+C\ln E+O(1), \tag{1}$$