[1]See, for example, M. L. Cohen and T. K. Bergstresser, Phys. Rev. 141, 789 (1966).
[2]R. A. Faulkner, Phys. Rev. 184, 713 (1969).
[3]W. Kohn and D. Schecter, Phys. Rev. 99, 1903 (1955); D. Schecter, J. Phys. Chem. Solids 23, 237 (1962).
[4]K. S. Mendelson and H. M. James, J. Phys. Chem. Solids 25, 729 (1964).
[5]C. Kittel and A. H. Mitchell, Phys. Rev. 96, 1488 (1954).
[6]W. Kohn and J. M. Luttinger, Phys. Rev. 97, 869 (1955).
[7]P. Fisher and H. Y. Fan, Phys. Rev. Lett. 2, 456 (1959); R. L. Jones and P. Fisher, J. Phys. Chem. Solids 26, 1125 (1965).
[8]A. Onton, P. Fisher, and A. K. Ramdas, Phys. Rev. 163, 686 (1967).
[9]R. J. Stirn and W. M. Becker, Phys. Rev. 148, 907 (1966); B. T. Ahlburn and A. K. Ramdas, Phys. Rev. 187, 932 (1969).
[10]P. J. Dean, C. J. Frosh, and C. H. Henry, J. Appl. Phys. 39, 5631 (1968).
[11]J. C. Phillips, Phys. Rev. B 1, 1540, 1545 (1970).
[12]J. M. Luttinger, Phys. Rev. 102, 1030 (1956).
[13]See, for example, A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton Univ., Princeton, N. J., 1960).
[14]P. Lawaetz, private communication.
[15]A. Baldereschi and N. O. Lipari, to be published.

Classical Analog of the Variable Moment of Inertia Formulas for Rotational States in Even-Even Nuclei*

P. Thieberger
Brookhaven National Laboratory, Upton, New York 11973
(Received 18 August 1970; revised manuscript received 16 November 1970)

A simple mechanical model has been found which gives the same angular velocity dependence of energy and angular momentum as postulated in the variable moment of inertia description of rotational states. The possibility for one of the two coefficients of this description to adopt negative values follows naturally from this model. Also the nonzero ground-state energies, which in some cases result from the variable moment of inertia description, are easily understood.

One explanation which has been advanced for the departure from the $J(J+1)$ energy formula for rotational states is the classical, rotation-dependent, elastic deformation (β stretching).[1] Much greater success has recently been obtained using the formulas

$$E=\tfrac{1}{2}\omega^2(\mathscr{I}_0+3C\omega^2), \qquad (1)$$

$$\hbar[J(J+1)]^{1/2}=\omega(\mathscr{I}_0+2C\omega^2)=\omega\mathscr{I}, \qquad (2)$$

where ω is the angular frequency, $\mathscr{I}$ is the moment of inertia, and $\mathscr{I}_0$ and C are constants. These expressions were first derived by Harris[2] from an extension of the cranking model,[3] and then shown to be equivalent to the variable moment of inertia (VMI) description.[4]

The striking success of these formulas in fitting the energy levels of ground-state rotational bands[4] was recently extended to an even wider range of nuclei by allowing the parameter $\mathscr{I}_0$ to become negative.[5,6] Thus, most, and perhaps all, even-even nuclei are now included in this description. The introduction of negative values of $\mathscr{I}_0$ also leads to an accurate prediction of a distinct discontinuity[5,6] in the so-called "Mallmann" plots,[7] the graphs of E_8/E_2 and E_6/E_2 versus E_4/E_2.

Harris's model, however, does not allow negative values for $\mathscr{I}_0$. Furthermore, the fact that a nonzero ground-state energy is obtained for some of the solutions[5,6] requires clarification. The purpose of the present work was to search for a classical model, similar to the centrifugal stretching model, which would be described not by the equations of that model but by Eqs. (1) and (2).

In the hydrodynamical model, on which the centrifugal stretching model is based, all the elements of mass in the nucleus take part in the motion contributing to the total angular momentum and energy. For real nuclei, however, only certain nucleons give rise to the total angular momentum while the rest, forming closed j shells, do not contribute. It seems, therefore, appropriate to search for a model in which the total mass M_T of a nucleus is divided into a rotating portion of mass M and a stationary part of mass M_T-M. In such a model the increase of the moment of inertia with angular momentum can be due either to a variation of the mass M or to a change of its orbit, or to a combination of both mechanisms. In the present approach the first possibility is chosen and it will therefore be assumed that the mass M rotates around an axis Z at a fixed radius R_0 with an angular frequency ω, and that variations of the moment of inertia arise solely from mass transfer between the two fractions.

In order to achieve states of equilibrium of such a system, restoring forces have to be postulated which must be overcome by the centrifugal acceleration for elements of mass to remain in the rotating part. Furthermore these restoring forces must increase as more mass is transferred to the outer portion if stable states of increasing angular momentum and energy are to correspond to increasing angular velocities. It will be sufficient to express this variation of the restoring forces with M in terms of the difference of potential energy dV necessary to transfer a mass dM from the central portion to the rotating portion. One of the simplest possible assumptions, i.e., a linear dependency expressed by

$$dV=(A+BM)dM \tag{3}$$

where A and B are constants, $B>0$, leads precisely to the VMI expressions (1) and (2) as will be shown below.

The mechanical model of Fig. 1 represents a system useful for visualizing the above-mentioned assumptions as well as the nature of the solutions that result.[8] It consists of liquid contained in two cylindrical reservoirs connected

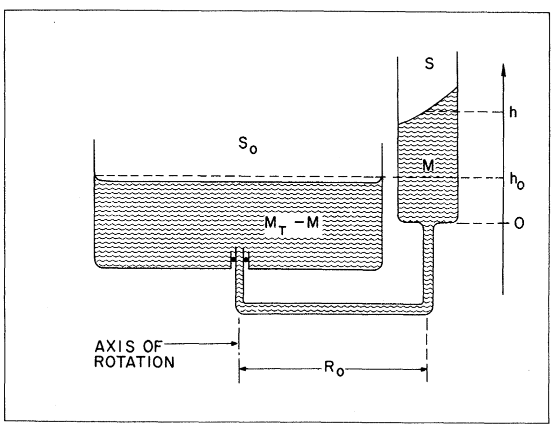

FIG. 1. Model used to visualize the assumptions and the results of the present approach. The large cylindrical reservoir, of cross-sectional area S_0, is stationary while the smaller one, of cross-sectional area S, rotates with an angular velocity ω. The liquid contained in these reservoirs can flow freely through the narrow tube that connects them.

by means of a thin tube. One of the reservoirs is stationary and the other one ratates around the first with angular velocity ω. Here the M-dependent potential energy is due to the gravitational potential difference for different fluid levels. In fact, neglecting the mass of the liquid contained in the connecting tube, the masses of the empty containers, and the centrifugal deformation of the liquid surface in the rotating reservoir, this system follows exactly the VMI expressions (1) and (2).

The constants of Eq. (3) can be expressed in this case as

$$A=-h_0 g(1+S/S_0), \tag{4}$$

$$B=(g/\rho)(1/S+1/S_0), \tag{5}$$

where g is the gravitational acceleration, ρ is the density of the liquid, S and S_0 are the cross-section areas of the cylindrical reservoirs, and h_0 is the height of the liquid surface when the angular frequency is zero.

The equilibrium condition for a given constant angular momentum $MR_0\omega$ corresponds to a minimum in the total energy of the system. Let $d\omega$ be the increment in angular velocity due to the transfer of the mass dM to the rotating portion at constant angular momentum. Then, in addition to the increment dV in potential energy, we shall have an increment dT of the kinetic energy given by

$$dT=d(\tfrac{1}{2}\omega^2 R_0^2 M)=\tfrac{1}{2}\omega^2 R_0^2 dM+M\omega R_0^2 d\omega. \tag{6}$$

Since the angular momentum is constant,

$$MR_0\omega=(M+dM)R_0(\omega+d\omega). \tag{7}$$

From (6) and (7) we then find

$$dT=-\tfrac{1}{2}\omega^2 R_0^2 dM. \tag{8}$$

The equilibrium condition can then be expressed, using Eqs. (3) and (8), as

$$dV+dT=(A+BM-\tfrac{1}{2}\omega^2 R_0^2)dM=0; \tag{9}$$

therefore,

$$M=(\tfrac{1}{2}\omega^2 R_0^2-A)/B \text{ for } \omega^2>2A/R_0^2, \tag{10}$$

or

$$M=0 \text{ for } \omega^2\leq 2A/R_0^2. \tag{10'}$$

The first case will obviously apply for any v~lue of ω if $A<0$ and such a situation is the one illustrated in Fig. 1 ($h_0>0$). If, however, the bottom of the rotating reservoir were higher than the level of the liquid at $\omega=0$ ($h_0<0, A>0$) then we would have an example of the second case for sufficiently small values of ω.

For a mass $M\neq 0$ at R_0 the moment of inertia will be

$$\mathscr{I}=MR_0^2=-\frac{A}{B}R_0^2+2\frac{R_0^4}{4B}\omega^2. \qquad (11)$$

The kinetic energy T is

$$T=\tfrac{1}{2}\omega^2R_0^2M=\tfrac{1}{2}\omega^2R_0^2(\tfrac{1}{2}\omega^2R_0^2-A)/B. \qquad (12)$$

The potential energy V is, calling M_0 the mass at R_0 for $\omega=0$ and using Eq. (3),

$$v=\int_{M_0}^{M}dV=(AM+BM^2/2)\big|_{M_0}^{M}. \qquad (13)$$

Since for $A<0$, $M_0=-A/B$, we obtain from Eqs. (10) and (13)

$$V=\omega^4R_0^4/8B. \qquad (14)$$

For $A>0$ we have instead $M_0=0$, and in this case

$$V=\omega^4R_0^4/8B-A^2/2B. \qquad (14')$$

From (12) and (14) we compute now the total energy E for the case in which $A<0$:

$$E=T+V=\tfrac{1}{2}\omega^2\left(-\frac{A}{B}R_0^2+3\frac{R_0^4}{4B}\omega^2\right). \qquad (15)$$

If we now compare Eqs. (11) and (15) with Eqs. (2) and (1), respectively, we see that they are identical if we identify

$$\mathscr{I}_0=-(A/B)R_0^2, \qquad (16)$$

$$C=R_0^4/4B. \qquad (17)$$

The constant energy $A^2/2B$ [see Eq. (14′)], which has to be subtracted from Eq. (15) in those cases for which $A>0$, accounts exactly and in a natural way for the nonzero values of the ground-state energies found in Refs. 5 and 6 for the cases in which $\mathscr{I}_0<0$.

It is now easy to interpret the meaning of a negative value of $\mathscr{I}_0$ ($A>0$) in terms of our model. It simply means that starting from a nondeformed nucleus ($M=0$) the first dM will already be subject to restoring forces such that its potential energy when transferred to the moving portion will be AdM. These are, therefore, nuclei which are not deformed in the ground state and the larger the value of A the larger the angular velocity which will be necessary to cause deformation.

For $A<0$ ($\mathscr{I}_0>0$) the equilibrium for $\omega=0$ is reached when $dV=0$ and $M_0=-A/B$. In this case the ground-state moment of inertia is $M_0R_0^2=\mathscr{I}_0$.

One aspect of Ref. 5 for which this model does not seem to provide a suitable interpretation is the so-called "$\alpha\beta$" solution used to describe closed-shell nuclei. This solution implies negative values of the moment of inertia of the 2^+ states and in terms of our model this would imply negative values for M. The fit obtained with these solutions is, however, so good that it seems justified to search for an interpretation of these negative masses, perhaps in terms of hole motion of some kind. Whether such an interpretation will be possible remains an open question.

The linear nature of Eq. (3) as well as the assumption of a constant radius of rotation should be considered as approximations which, probably, work as well as they do because of the fact that variations in the amount of mass rotating at a radius R_0 are relatively small fractions of the total nuclear mass. For instance, in all cases the rotating fractions necessary to account for the values of the moment of inertia given by the VMI model[4,6] for spins up to 10 are less than 26% of the total mass.

The deviations from the VMI predictions recently observed[9] as almost equidistant high-spin states in ^{160}Dy would be interpreted in terms of the present approach as a leveling off in the increase of potential energy [Eq. (3)] rather than to a phase transition leading to a rigid-rotor value for the moment of inertia.

I wish to thank W. Gelletly and M. Mariscotti for stimulating discussions and suggestions. I am grateful to E. Church, E. der Mateosian, J. O. Rasmussen, A. W. Sunyar, and J. Weneser for encouraging comments and criticisms.

---

*Work performed under the auspices of the U. S. Atomic Energy Commission.

[1]R. M. Diamond, F. S. Stephens, and W. T. Swiatecki, Phys. Lett. 11, 315 (1964).

[2]S. H. Harris, Phys. Rev. 138, B509 (1965).

[3]D. R. Inglis, Phys. Rev. 96, 1059 (1954).

[4]M. A. J. Mariscotti, G. Scharff-Goldhaber, and B. Buck, Phys. Rev. 178, 1864 (1969).

[5]M. A. J. Mariscotti, Phys. Rev. Lett. 24, 1242 (1970).

[6]G. Scharff-Goldhaber and A. S. Goldhaber, Phys. Rev. Lett. 24, 1349 (1970).

[7]C. A. Mallmann, Phys. Rev. Lett. 2, 507 (1959).

[8]The idea of using this type of model to illustrate the present approach is due to J. O. Rasmussen, private communication.

[9]A. Johnson, H. Ryde, and J. Sztarkier, to be published.