first sheet if $m > 2$.

[12]A. Jaffe, Commun. Math. Phys. 1, 127 (1965).

[13]See, e.g., G. Hardy, *Divergent Series* (Oxford Univ., Oxford, England, 1949).

[14]We regard the fact that the energy level is uniquely determined by the asymptotic series as the crucial result of this note. For both Padé and Borel summability, the required uniqueness follows from Carleman's criterion: If $|f(x)| < a_n |z|^n$ for all z in a region $\{z \mid |\arg z| < \frac{1}{2}\pi |z| < B\}$ where f is analytic, and if

$$\sum_{n=1}^{\infty} a_n^{-1/n} = \infty,$$

then $f = 0$. Thus the unique determination from the series follows from our strong bound on the remainder.

[15]T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1966).

[16]That V can be so realized is the content of the spectral theorem; a particular such realization on "Q space" is basic to some results in Refs. 8 and 9.

[17]That $V = V_+ - V_-$ obeys $V < a'N^2 + b'$ is an "elementary N_τ estimate" of the type employed by Glimm and Jaffe; it depends on the fact that V is a sum of Wick-ordered products of four creation and annhilation operators integrated with square integral kernels. That $2V_- < a''N + b'' < a''N^2 + b''$ is a result of the type first proven by Nelson (Ref. 7) and simplified by Segal (Ref. 9).

[18]That $\|(N+1)^{-1} V_j (N+1)^{-1}\|$ is finite is the N_τ estimate referred to in note 17 above. Since $(N+1)(H_0 - \lambda)^{-1}$ acts as $(n+1)/\sum_{j=1}[w(k_i) - \lambda]$ on the n-particle space with $|\lambda| = \frac{1}{2}m$ and each $w(k_i) > m$, $\|(N+1)(H_0 - \lambda)^{-1}\| <$ some constant independent of $\arg\lambda$.

Predictions for High-Energy Elastic and Inelastic Scatterings in φ^3 Theory*

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We study high-energy elastic and inelastic processes in a φ^3 theory based on the s-channel iterations of t-channel ladder diagrams. The main results are the following: (a) The total cross section goes to zero, a constant, or $\sim(\ln s)^2$ for the coupling constant being smaller than, equal to, or larger than a critical value. (b) Inelastic differential cross sections are computed and the s-channel unitarity is explicitly verified. (c) One-particle spectrum, multiplicity, number distribution, etc. are presented. The implications of these results to hadron physics are discussed.

Because of lack of a better alternative to the conventional perturbation expansion, quantum field theory has not been proved to be useful in the analysis of high-energy behavior of elastic and inelastic scatterings in strong interactions. As the recently developed infinite-momentum technique permits one to handle the leading high-energy behavior of a very wide class of Feynman diagrams,[1-4] it may be hoped that study of certain field-theory models will at least reveal some general qualitative features concerning the questions mentioned above. Much work along this line has been done by many authors.[3-5] On the basis of their results of high-order calculations of the elastic-scattering amplitude in massive quantum electrodynamics (QED),[3] Cheng and Wu recently made a number of predictions on the elastic-scattering amplitude, the differential and integrated elastic cross sections, and the total cross section for hadron-hadron scattering at infinite energy. One may hope that a simple model with φ^3 coupling will also give the same qualitative predictions as QED; furthermore, the simplicity of the model allows one to draw more physical consequences. In this Letter we report some predictions of the simple model with a φ^3 coupling for elastic and inelastic hadron scattering at very high energies.

Our model is defined as follows: For the elastic-scattering amplitude we first summed the leading terms in each order of perturbation of the t-channel straight ladders plus those obtained by interchange of the Mandelstam variables s and u. We then performed the s-channel iteration

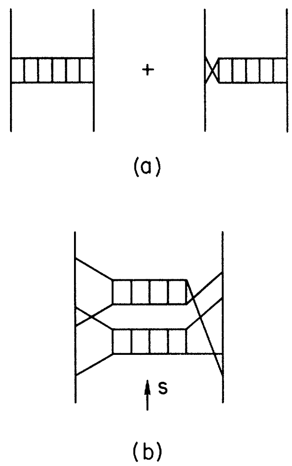

FIG. 1. (a) t-channel ladders. (b) Example of s-channel iteration of t-channel ladders in (a); permutation among the legs of all ladders is understood.

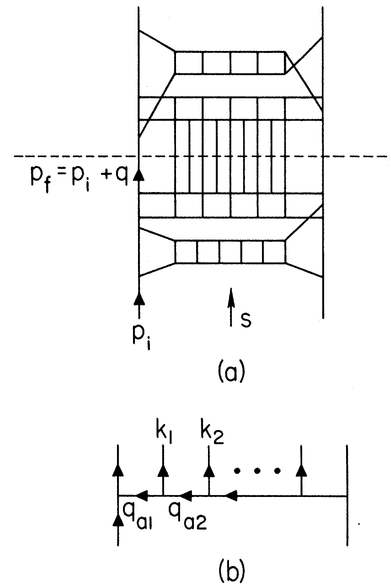

FIG. 2. (a) Example of unitarity diagrams for a general production process in which two ladders are opened. The solid lines cut by the dashed line correspond to the real final particles. On each side of the dashed line, permutation among the legs of open and unopen ladders is understood (b) Kinematics of an open ladder in detail.

with the t-channel ladders as single units. We are only interested in the physical regions in which the incident particles retain their large momenta; i.e., the legs of the exchange ladders do not carry any appreciable fraction of the large incident momenta. This procedure leads to the eikonalization of the t-channel ladders.[3,4] For the inelastic production amplitude we find that the dominant contribution comes from diagrams with any number of totally open ladders modified by unopen ones. The diagrams contributing to the elastic- and inelastic-scattering amplitudes are shown in Figs. 1 and 2, respectively. The main results of our calculations are the following:

(a) The total cross section has completely different high-energy behavior for the weak-coupling, the critical-coupling, and the strong-coupling cases according to whether $g^2/16\pi^2\mu^2-2$ is less than, equal to, or greater than zero. (The coupling constant g is defined by $\mathfrak{L}_I = \frac{1}{6}g\varphi^3$.) In the first case, the Born term dominates at large energy, and the cross section goes to zero as $s \to \infty$. In the second case, the total cross section approaches a constant. In the third case, however, the total cross section increases like $(\ln s)^2$ at large s, i.e., it saturates the Froissart bound. Since the first case is rather trivial, we shall not discuss it in detail.

(b) The total, elastic, and inelastic cross sections σ_T, σ_E, and σ_I are the same as those obtained from an s-dependent absorption model:

$$\sigma_T = 2\int d^2b[1-e^{-A(s,\vec{b})}], \tag{1}$$

$$\sigma_E = \int d^2b[1-e^{-A(s,\vec{b})}]^2, \quad \sigma_I = \int d^2b[1-e^{-2A(s,\vec{b})}], \tag{2}$$

with $\vec{b}$ the impact parameter and $A(s,\vec{b})$ an opaqueness given later. At large energy, and for the strong-coupling case, these cross sections reduce to

$$\sigma_E = \sigma_I = \tfrac{1}{2}\sigma_T = \pi b_{\max}{}^2, \tag{3}$$

where $b_{\max}$ is found to be proportional to $\ln s$. These results agree with those in QED.[3] For the critical case, only a single ladder contributes at $s=\infty$. It leads to a constant σ_T and $\sigma_E \propto 1/\ln s$.

(c) The inelastic processes are those shown in Fig. 2. From the differential inelastic cross sections for multiparticle states, we can compute the one-particle spectrum, number and energy distributions, etc. The one-particle spectrum for a detected particle with four-momentum $(k^0, \vec{k})$ is given by

$$d\sigma = \frac{g^6}{4}\exp\left[\left(\frac{g^2}{16\pi^2\mu^2}-2\right)\left(\frac{g^2}{8\pi^2\mu^2}+1\right)^{-1}\ln s\right]\int\frac{d^2q}{(2\pi)^2}\,\frac{1}{(\vec{q}^{\,2}+\mu^2)^2[(\vec{k}+\vec{q})^2+\mu^2]^2}\,\frac{d^3k}{(2\pi)^3 2k^0}. \tag{4}$$

The s-dependent factor drops out for the critical case. The one-particle spectrum leads to a dx/x distribution in the longitudinal momentum, and an s-independent transverse-momentum distribution.

(d) The number distribution function $P(n)$ for the strong-coupling inelastic processes is a superposition of Poisson distributions in the impact space:

$$P(n) = (\pi b_{\max}{}^2)^{-1} \int d^2b\, e^{-B(s,\vec{b})} B(s,\vec{b})^n/n!, \tag{5a}$$

where $B(s,\vec{b})$ is related simply to the opaqueness $A(s,\vec{b})$, and has the same size $b_{\max}$. Equation (5a) has a longer tail at large n than a single Poisson distribution. For critical coupling, the number distribution is simply a Poissonian,

$$P(n) = (2\ln s)^n/(s^2 n!). \tag{5b}$$

We shall only outline briefly our calculation. The details of the calculations will be published elsewhere. First, the sum of leading $\ln s$ terms in a t-channel ladder is well known. It leads to a structure of a Regge pole[6]:

$$T_L = -i\pi g^2[\alpha(\vec{k}^2)+1]s^{\alpha(\vec{k}^2)}, \tag{6}$$

$$\alpha(\vec{k}^2)+1 \equiv \frac{g^2}{4\pi}\int \frac{d^2q}{(2\pi)^2} \frac{1}{[(\vec{q}+\frac{1}{2}\vec{k})^2+\mu^2][(\vec{q}-\frac{1}{2}\vec{k})^2+\mu^2]}, \tag{7}$$

where $\vec{k}$ is the momentum transfer with components only in the 1, 2 plane. The s-channel iteration of these ladders in the manner described previously gives the standard eikonal form

$$T(s,k^2) = -i2s\int d^2b\, e^{-i\vec{k}\cdot\vec{b}}[1-e^{-A(s,\vec{b})}], \tag{8}$$

where $A(s,\vec{b})$ in our model is given by

$$A(s,\vec{b}) = \frac{\pi g^2}{2s}\int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}\cdot\vec{b}}[\alpha(\vec{k}^2)+1]s^{\alpha(\vec{k}^2)}. \tag{9}$$

The "potential" given by (9) is purely imaginary. Thus our model is purely absorptive and the elastic amplitude is dominated by its imaginary part. Equations (1) and (2) then follow from (8). As $s\to\infty$, $A(s,\vec{b})$ behaves asymptotically like $(s^{\alpha(0)-1}/\ln s)\exp(-\mathrm{const}\,\vec{b}^2/\ln s)$. For $\alpha(0)-1 = g^2/16\pi^2\mu^2-2<0$, $A(s,\vec{b})$ vanishes at large s, as does the forward-scattering amplitude. However, when $g^2/16\pi^2\mu^2-2>0$, $A(s,\vec{b})$ acquires an increasing s dependence. Hence, the strength of the absorptive potential increases as a power of s. According to the original argument of Froissart,[7] the effective range for such a potential should increase as $\ln s$. In our model, the range $b_{\max}$ is

$$b_{\max}{}^2 = \frac{g^2}{24\pi^2\mu^4}\left(\frac{g^2}{16\pi^2\mu^2}-2\right)(\ln s)^2 \tag{10}$$

near $g^2/16\pi^2\mu^2=2$, and a somewhat different coefficient for $(\ln s)^2$ for large g^2. Since $A(s,\vec{b})$ varies rapidly at large s from infinity to zero as we increase $b=|\vec{b}|$ passing through $b_{\max}$, we have $e^{-A(s,\vec{b})} = \theta(b-b_{\max})$ where $\theta(x)$ is the step function, from which Eq. (3) follows. If $\alpha(0)=1$, A is small and $1-e^{-A}\approx A$. Hence σ_T is constant and $\sigma_E \propto 1/\ln s$.

We now turn to the more interesting case of production processes. First of all, we find that the dominant production mechanism is given by the diagrams shown in Fig. 2. Namely, given the elastic amplitude corresponding to Fig. 1, the final states produced by the unitarity cut are those obtained by cutting through every rung of any number of the ladders, as in Fig. 2(a). This can be easily understood since the momenta of the lines connecting the rungs are spacelike and therefore these lines cannot be realized as final particles. The lines at the far right and far left carry the large longitudinal momentum $\pm P$ supplied by the two initial particles, while the other lines associated with each ladder have only a very small fraction of the longitudinal momentum P.

The amplitude for an inelastic process can be written straightforwardly in the impact-parameter representation. The inclusion of unsplit ladders leads to a factor $e^{-A(s,\vec{b})}$, just as in the elastic scattering. The amplitude can be written as

$$T(q) = 2is\int d^2b\, e^{-i\vec{q}\cdot\vec{b}} e^{-A(s,\vec{b})} \prod_{(\text{all open ladders})}\left[\frac{ig}{2s}\int\frac{d^2q_{a1}}{(2\pi)^2}e^{i\vec{q}_{a1}\cdot\vec{b}}\prod_i \frac{g}{q_{ai}{}^2+\mu^2}\right]. \tag{11}$$

To obtain the contribution to the inelastic cross section, we square the matrix element (11) multiplied by the overall energy-momentum–conserving delta function and integrate over the phase space. The two halves of each open ladder rejoin to give simple results. In this way we obtain the partial cross section for N ladders open by the unitarity cut,

$$\sigma_I^{(N)} = (1/N!)\int d^2b\, e^{-2A(s,\vec{b})}[2A(s,\vec{b})]^N, \tag{12}$$

which sums up to the total inelastic cross section in (2). The attenuation factor $e^{-A(s,\vec{b})}$ in (11) ensures that inelastic contributions are always finite and consistent with Froissart bound. If we calculate the cross section for a particular channel of n soft-particle emission, then we get the particle number distributions $P(n)$ as given by (5) as $s\to\infty$.

If we leave the momentum of one soft particle unintegrated in calculating σ_I, we obtain the one-particle spectrum

$$d\sigma = \frac{g^6}{4}s^{\alpha(0)-1}\int\frac{d^2q}{(2\pi)^2}\frac{1}{(\vec{q}^{\,2}+\mu^2)^2[(\vec{k}+\vec{q})^2+\mu^2]^2}\frac{d^3k}{(2\pi)^3 2k^0}. \tag{13}$$

From (13) or (5b) we obtain an average multiplicity $\propto \ln s$ for critical coupling. For strong coupling the multiplicity is

$$\langle n\rangle = (c/\ln s)s^{\alpha(0)-1} \tag{14}$$

with c a constant and $\alpha(0) = g^2/16\pi^2\mu^2 - 1$. This result has a serious defect. As the coupling becomes very strong, the multiplicity may increase faster than the ultimate limit $\sqrt{s}$ allowed by energy conservation. The inconsistency can be traced back to the inadequacy of our approximation which neglects the energies of the soft particles in the energy-conservation condition. The $(\ln s)^n$ factors which eventually exponentiate to a power of s come from the longitudinal phase-space integrals

$$2\int_0^{\sqrt{s}} dk_{\parallel}/k^0 \approx \ln s. \tag{15}$$

The upper limit is only an order-of-magnitude estimate. A more reasonable cutoff is to assume that the upper limit is most probably given by the average energy shared by each particle $s^{1/2}/\langle n\rangle$. If $\langle n\rangle$ grows as a power of s, the correction becomes significant. To correct this error, we propose a self-consistent physical procedure. Let us assume that, within insignificant logarithmic correction,

$$\langle n\rangle \propto s^a, \tag{16}$$

where a is a parameter to be determined. This procedure will affect our previous results by the substitution

$$\ln s \to (1-2a)\ln s \tag{17}$$

in the elastic and inelastic amplitudes. The same cutoff is to be applied to every particle, virtual or real, so that general principles, such as unitarity, can be maintained. Equation (14) is modified to

$$\langle n\rangle_c = \frac{c}{\ln s}s^{\alpha(0)(1-2a)-(1+2a)} = \frac{c}{\ln s}s^a, \tag{18}$$

since we demand that this expression be consistent with the original assumption (16). This gives

$$a = \frac{(g^2/16\pi^2\mu^2)-2}{(g^2/8\pi^2\mu^2)+1} = \frac{\alpha(0)-1}{2\alpha(0)+3}. \tag{19}$$

As $g^2\to\infty$, a approaches the limit $\frac{1}{2}$, no longer in conflict with energy conservation.

In the following we further discuss the results obtained above. Our proposed cutoff procedure has the interesting consequence that the total cross section cannot grow indefinitely with increasing coupling constant. It approaches a limit $\sigma_T(g^2\to\infty)\approx\frac{5}{3}(\pi/\mu^2)(\ln s)^2$. An increasing multiplicity such as (18) is not inconsistent with experimental data available. Equation (4) shows dx/x distribution for the longitudinal momentum of an emitted soft particle. Feynman's scaling hypothesis[8] for the one-particle spectrum is satisfied for critical coupling, but is violated for strong coupling by the presence of the factor s^a. The violation will be weak if a is small as suggested by a slowly increasing multiplicity observed empirically. The possibility of a weak s dependence in the one-particle spectrum should be looked for experimentally.

However, the transverse momentum spectrum predicted by (4) is in contradiction to available data which all indicate a Gaussian or exponential falloff for the transverse momenta. This slow falloff for the transverse momentum is due to the lack of a long-chain correlation and the pointlike vertices in this model.

In conclusion, we make the following remarks:

(1) The direct verification of unitarity as expressed by $\sigma_E + \sum_{\text{inel}} \sigma_I^{(N)} = \sigma_T$ is an important consistency check for the treatments of elastic and inelastic scatterings. Since a large fraction of the total events in high-energy hadron scattering are inelastic, a theoretical handling of these inelastic states consistent with that for elastic ones is desirable. Our treatment here perhaps is a meaningful beginning for such a theoretical development.

(2) Our results in terms of a single impact-parameter representation very likely are due to our neglect of fragmentation processes.[9] In a more general, though not necessarily perfect, approach we may picture that both the colliding particles first dissociate into fragments and the fragments from the target scatter with those from the projectile. Our present calculation presumably applies only to the individual scatterings between the fragments.

(3) The multiplicity difficulty encountered in this study is a warning of the danger hidden in this approach of summing leading terms. This is particularly so because the inconsistency cannot be easily detected in the gross results such as the elastic amplitude and elastic and total cross sections.

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Note added in proof. – After submitting this paper, we were informed that similar results were also obtained by Hasslacher et al.[10]

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