Mass Dispersion Relations in the Light of the Light Cone*

Richard A. Brandt†
University of Maryland, College Park, Maryland 20742, and Rockefeller University, New York, New York 10021

and

Giuliano Preparata‡
Brookhaven National Laboratory, Upton, New York 11973, and Rockefeller University, New York, New York 10021
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Operator-product expansions near the light cone are used to study the convergence and saturation properties of some mass dispersion relations. We develop a general formalism and apply it to justify partial conservation of axial-vector current and to correct vector-meson dominance. We specifically treat the K_{l3} form factors and the decays $\pi^0 \to \gamma\gamma$, $\omega \to \pi\gamma$, and $\omega \to 3\pi$.

In the last few years, a considerable effort has been devoted to attempting to understand the nature of processes involving the interaction of a massive photon with hadrons. The Stanford Linear Accelerator Center (SLAC) deep-inelastic electron-proton scattering experiments[1] have strongly suggested that, although many finer details depend on purely hadron structure, a major feature of these processes, the so-called "scaling behavior," is also common to "structureless" photon-lepton scattering. It is generally believed that such experiments test the short-distance behavior of the hadrons by probing them with massive photons (acting like an extremely high-resolution microscope). Attempts have been made to give this idea a quantitative content, and, most notably, the restrictions implied by local current algebra have been used by Bjorken[2] to obtain a number of interesting results.

It has also been shown[3,4] that, in the SLAC experiment[1] and in the recent Columbia-Brookhaven National Laboratory(BNL) experiment[5] measuring massive μ-pair production, the more intricate behavior near the light cone, rather than simply at equal times, is relevant for understanding the observed results. An operator-product expansion near the light cone has been shown by us to be valid under quite general circumstances, including renormalized perturbation theory,[6] and has proved to be particularly suited for treating the general class of processes previously mentioned. The purpose of this paper is to show that such light-cone expansion, together with the observation that the leading light-cone behavior sets in very quickly, can be used to determine the convergence properties of mass dispersion relations and to decide the extent to which low-mass saturation of these relations is reliable.

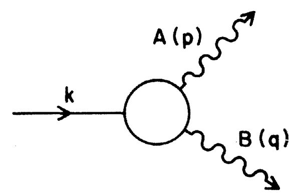

FIG. 1. Kinematics of the three-point function $A(p^2, q^2)$.

Consider a vertex function with kinematics as shown in Fig. 1, where the scalar particle (k) is on shell ($k^2 = m^2$) and p and q ($k = p + q$) are the momenta carried by two scalar currents $A(x)$ and $B(x)$:

$$A(p^2, q^2) = \int d^4x\, e^{-ip\cdot x}\langle 0|T[A(x)B(0)]|k\rangle. \tag{1}$$

It is easy to see that the behavior of $A(p^2, q^2)$ in the limit $p^2 \to \infty$ with $\omega \equiv p^2/2\nu \equiv [1-(q^2-m^2)/p^2]^{-1}$ fixed (∞ included) is determined by the behavior of $A(x)B(0)$ near (namely, within $1/p^2$) the light cone $x^2 = 0$[6]. Thus, to compute this limit, we can use the general light-cone expansion[6]

$$A(x)B(0) \sim E(x^2 - i\epsilon x_0)\sum_n x^{\alpha_1}\cdots x^{\alpha_n} \mathcal{O}_{\alpha_1\cdots\alpha_n}{}^{(n)}(0), \tag{2}$$

where the $\mathcal{O}_{\alpha_1\cdots\alpha_n}{}^{(n)}(0)$ are local operators. Defining

$$\sum_n x^{\alpha_1}\cdots x^{\alpha_n}\langle 0|\mathcal{O}_{\alpha_1\cdots\alpha_n}{}^{(n)}(0)|k\rangle = f(k\cdot x) + \mathcal{O}(x^2), \tag{3}$$

we obtain

$$A(q^2, p^2) \to \int d^4x\, e^{-ip\cdot x}E(x^2 - i\epsilon x_0)f(k\cdot x). \tag{4}$$

Thus the general singularity $E(z)=z^{-r}$ gives the result

$$A(p^2,q^2)\to -\frac{e^{\frac{1}{2}i\pi r}(2\pi)}{\Gamma(r)}(p^2)^{r-2}\omega^{2-r}F_r(\omega), \tag{5}$$

where

$$F_r(\omega)\equiv \int_0^\infty d\lambda\, e^{i\omega\lambda}\lambda^{1-r}f(\lambda). \tag{6}$$

Two interesting special cases are the limit $\omega\to 1$ [so that $(q^2/p^2)\to 0$ and the limit is $p^2\to\infty$ with q^2 fixed], in which

$$A(p^2,q^2)\xrightarrow[q^2\ \text{fixed}]{p^2\to\infty}(p^2)^{r-2}F_r(1), \tag{7}$$

and the limit $\omega\to\infty$ [so that $(q^2/p^2)\to 1$ and the limit is the Bjorken[2] limit $p_0\to\infty$ with $\vec{p}$ fixed], in which

$$A(p^2,q^2)\xrightarrow[\vec{p}\ \text{fixed}]{p_0\to\infty}(p_0)^{2r-4}f(0). \tag{8}$$

As expected, the limit (8) is controlled by the first nonvanishing (although perhaps infinite) equal-time commutator as determined by (2).

In simple perturbation theories, one finds $r=1$ (within logs) and $f(\lambda)\sim e^{i\lambda}$ so that $F_1(\omega)$ has a pole at $\omega=1$ and (7) becomes meaningless, the correct behavior being $A\to$ const. We explicitly assume that our $F_r(\omega)$ do not develop such poles. This assumption accounts for the observed rapid decrease of empirical form factors (see below) and the smooth behavior of the structure functions measured at SLAC and amounts to assuming a composite structure for the hadrons. It is, presumably, the same mechanism which Reggeizes the fixed poles of perturbation theory that eliminates the poles in $F_r(\omega)$.

We shall make a second assumption in order to determine the values of r relevant in specific cases. We assume that all relevan field and current operators have the same (canonical) dimensions that they have in the gluon model (triplet quarks coupled to a massive neutral vector meson via the baryon number current) (ignoring logs). The gluon model thus treated has been very successful in accounting for many aspects of processes like the ones we are considering,[7] and this specific assumption gives the essentially unique singularity structure for electromagnetic currents consistent with the SLAC and Columbia-BNL experiments.[6]

Our final assumption will be that asymptotic behavior sets in quite quickly, namely for $p^2\sim 2$ BeV^2. This assumption is strikingly supported by the results of SLAC and Columbia-BNL. Its implications for our purposes are that (7) becomes valid for $p^2\geqslant 2$ BeV^2 and that $f(\lambda)$ has support concentrated very near $\lambda=0$. This last statement accounts for the rapid approach of the electroproduction scaling function to its (constant) asymptotic limit. It means, in particular, that $F_1(1)$ is of the order of $f(0)$.

We proceed to apply these ideas to discuss mass dispersion relations. The amplitude $A(p^2,q^2)$ is assumed to be analytic in the cut p^2 plane, with a cut starting at $p^2=\alpha>0$. We can, therefore, write the "finite-mass dispersion relation"

$$A(p^2,q^2)=\frac{1}{\pi}\int_\alpha^\Lambda dp'^2\frac{a(p^2,q^2)}{p'^2-p^2+i\epsilon}+\frac{1}{2\pi i}\int_{c_\Lambda}dp'^2\frac{A(p'^2,q^2)}{p'^2-p^2+i\epsilon}, \tag{9}$$

where $a(p^2,q^2)$ is the absorptive part (in p^2) of $A(p^2,q^2)$ and c_Λ is the circular contour $|p^2|=\Lambda$. For $\Lambda>2$ BeV^2, we thus obtain

$$A(p^2,q^2)\simeq\frac{1}{\pi}\int_\alpha^\Lambda dp'^2\frac{a(p'^2,q^2)}{p'^2-p^2+i\epsilon}+\frac{F_r(1)}{2\pi i}\int_{c_\Lambda}dp'^2\frac{(p'^2)^{r-2}}{p'^2-p^2+i\epsilon}. \tag{10}$$

Integrating A over c_Λ, we get the further useful relation

$$0=\frac{1}{\pi}\int_\alpha^\Lambda dp'^2 a(p'^2,q^2)+\frac{F_r(1)}{2\pi i}\int_{c_\Lambda}dp'^2(p'^2)^{r-2}. \tag{11}$$

We are thus paralleling the "finite-energy sum rule" treatment of four-point functions. The important fact that Λ can be as small as 2 BeV^2 is analogous to the usefulness of the concept of "duality."

Let us suppose that there is a low-lying particle of mass μ with the quantum numbers of $A(x)$ so that

$$a(p^2,q^2)=\pi\delta(p^2-\mu^2)a_p(q^2)+a_n(p^2,q^2). \tag{12}$$

Then, (10) and (11) become (canonical dimensionality implies that r is an integer)

$$A(p^2,q^2)=\frac{a_p(q^2)}{\mu^2-p^2+i\epsilon}+\frac{1}{\pi}\int_\alpha^\Lambda dp'^2\frac{a_n(p'^2,q^2)}{p'^2-p^2+i\epsilon}+\delta_{r,2}F_r(1) \tag{13}$$

and

$$0=a_p(q^2)+1/\pi\int_\alpha^\Lambda dp'^2 a_n(p'^2,q^2)+\delta_{r,1}F_r(1). \tag{14}$$

We first apply the above formalism to study the momentum dependence of the off-shell K_{l3} form factors.[8] We consider the amplitude

$$D(p^2,q^2)=\frac{(2)^{1/2}(m_\pi{}^2-p^2)}{f_\pi m_\pi{}^2}\int d^4x\, e^{ip\cdot x}\langle 0|T[D^3(x)\mathfrak{D}^K(0)]|K^+,k\rangle,$$

where $D^a=\partial^\mu A_\mu{}^a$ and $\mathfrak{D}^a=\partial^\mu V_\mu{}^a$, in terms of the vector and axial-vector currents. The relevant light-cone expansion is

$$D^3(x)\mathfrak{D}^K(0)\to(x^2-i\epsilon x_0)^{-2}\sum_{n=1}^\infty x^{\alpha_1}\cdots x^{\alpha_n}\mathcal{O}_{\alpha_1\cdots\alpha_n}{}^{(n)}(0)+0(1/x^2). \tag{15}$$

This follows from the gluon-model results $\dim D=\dim\mathfrak{D}=3$, $\dim\mathcal{O}^{(n)}=2+n$ (the minimal dimension for an odd-parity field is three). In particular, $\mathcal{O}_\alpha{}^{(1)}(0)\propto\epsilon_2\epsilon_3 A_\alpha{}^K(0)$, where $\epsilon_2(\epsilon_3)$ measures the amount of SU(2)⊗SU(2) [SU(3)] symmetry breaking.[8] We obtain as above

$$D(q^2,p^2)\underset{q^2\text{ fixed}}{\xrightarrow{p^2\to\infty}}\delta+O(1/p^2), \tag{16}$$

where $\delta\sim\int_0^\infty d\lambda\, e^{i\lambda}f(\lambda)\lambda^{-1}=\text{const}$. Now, since the sum in (15) starts at $n=1$, we have $f(\lambda)=\lambda g(\lambda)$ and $g(0)\propto\epsilon_2\epsilon_3 f_k$. Since $g(\lambda)$ is assumed to have support near $\lambda=0$, we conclude that $\delta=O(\epsilon_2\epsilon_3 f_k)$.

Equations (13) and (14) then become

$$D(0,q^2)\simeq\frac{1}{(2)^{1/2}}[(m_K{}^2-m_\pi{}^2)f_+(q^2)+q^2f_-(q^2)]+\frac{1}{\pi}\int_{9m_\pi{}^2}^\Lambda dp'^2\frac{d(p'^2,q^2)}{p'^2+i\epsilon}+O(\epsilon_2\epsilon_3), \tag{17}$$

$$0\simeq\frac{m_\pi{}^2}{(2)^{1/2}}[(m_K{}^2-m_\pi{}^2)f_+(q^2)+q^2f_-(q^2)]+\frac{1}{\pi}\int_{9m_\pi{}^2}^\Lambda dp'^2 d(p'^2,q^2)+O(\epsilon_2\epsilon_3) \tag{18}$$

where $f_\pm(q^2)$ are the K_{l3} form factors, d is the absorptive part of D, and the $O(\epsilon_2\epsilon_3)$ terms arise as above from the $O(x^{-2})$ of each term in (15). Using[9]

$$\int_{9m_\pi{}^2}^\Lambda dp^2\, d(p^2)/p'^2\simeq\int_{9m_\pi}^2 dp^2\, d(p^2)/9m_\pi{}^2,$$

we obtain from (17) and (18)

$$D(0,q^2)\simeq 1/(2)^{1/2}[(m_K{}^2-m_\pi{}^2)f_+(q^2)+q^2f_-(q^2)]+O(\epsilon_2\epsilon_3). \tag{19}$$

Using finally the low-energy theorem

$$D(0,m_K{}^2)=\frac{1}{(2)^{1/2}}m_K{}^2\frac{f_K}{f_\pi}\frac{\epsilon_3}{\epsilon_3\frac{2}{3}\epsilon_2},$$

we can justify either strong partial conservation of axial-vector current (PCAC) ($\epsilon_2\ll\epsilon_3$, $\xi\equiv f_-/f_+\sim 0$) or weak PCAC ($\epsilon_3\ll\epsilon_2$, $\xi\sim 1$). These are precisely the conclusions of Ref. 8, now made more compelling because of the smallness of Λ.

We can similarly show how pion poles dominate physical form factors. We obtain asymptotic behavior of the form $A(p^2,q^2)\to a+b/p^2$ with a and b constant so that when we go on shell by taking $\lim_{q^2\to M^2}(q^2-M^2)A(p^2,q^2)\equiv A(p^2)$, we obtain $p^2A(p^2)\to 0$. As in Ref. 8, this leads to the superconvergence universality result

$$A(0)\simeq A_\pi(1-m_\pi{}^2/\overline{m}^2), \tag{20}$$

where $9m_\pi{}^2\lesssim\overline{m}^2\lesssim\Lambda\simeq 2\text{ BeV}^2$. These results support the use made of weak PCAC in Ref. 8.

We next apply similar considerations to study vector-meson dominance (VMD) as applied to the related processes $\pi^0\to\gamma\gamma$, $\omega\to\pi\gamma$, and $\omega\to 3\pi$. We define the $\pi^0\to\gamma\gamma$ amplitude in terms of the $\pi^0\to\gamma^3(k_1)$

$+\gamma^8(k_2)$ one $A(k_1k_2)$:

$$(2\sqrt{3})A(k_1, k_2) = \epsilon_{\mu\nu\alpha\beta}\epsilon^\mu(k_1)\epsilon^\nu(k_2)k_1^{\alpha}k_2^{\beta}F(k_1^2, k_2^2). \tag{21}$$

The relevant operator product expansion is[6]

$$J_\mu^3(x)J_\nu^8(0) \to \epsilon_{\mu\nu\alpha\beta}(x^2-i\epsilon x_0)^{-2}x^{\alpha}\sum_{n=0}^{\infty} x^{\alpha_1}\cdots x^{\alpha_n}\mathcal{O}_{\alpha_1\ldots\alpha_n}{}^{\beta,(n)}, \tag{22}$$

where the omitted terms do not contribute to (21). In the gluon model $\mathcal{O}_\beta{}^{(0)}(0) \propto A_\beta{}^3(0)$. As above we obtain[10]

$$F(k_1^2, k_2^2) \underset{\substack{k_1^2\to\infty \\ k_2^2 \text{ fixed}}}{\longrightarrow} \frac{1}{k_1^2}F_1(1) \sim \frac{1}{k_1^2}\frac{2f_\pi e^2}{3(2)^{1/2}}, \tag{23}$$

$$f(0,0) \simeq (e/\gamma_\omega)A(\omega\to\pi\gamma) + A_c/m_c^2, \tag{24}$$

and

$$\tfrac{1}{3}\sqrt{2}f_\pi \simeq (em_\omega^2/\gamma_\omega)A(\omega\to\pi\gamma) + A_c, \tag{25}$$

where $em_\omega^2/2\gamma_\omega(m_\omega^2) = em_\omega^2/2\gamma_\omega$ is the on-shell ω-γ junction and A_c and m_c represent an average of the continuum effects so that we expect $m_\omega^2 \leqslant m_c^2 \leqslant \Lambda = 2\ \mathrm{BeV}^2$. Equations (24) and (25) give

$$F(0,0) \simeq (e/\gamma_\omega)(1-m_\omega^2/m_c^2)A(\omega\to\pi\gamma) + 2^{1/2}f_\pi e^2/3m_c^2. \tag{26}$$

Note that the first term in (26) is the usual Gell-Mann–Sharp-Wagner (GMSW) term,[11] but with a correction factor $(1-m_\omega^2/m_c^2)$, and the second term comes from the light-cone behavior. We shall compare this prediction with experiment below.

We call $A(k_1^2)$ the off-shell $\omega\to\pi\gamma$ invariant amplitude so that the Feynman amplitude is $A(0)\epsilon_{\mu\nu\alpha\beta}\epsilon^\mu(k)\epsilon^\nu(k_1)k_1^{\alpha}k_2^{\beta}$. We obtain as above [cf. Eq. (20)]

$$A(0) \simeq g_{\omega\rho\pi}(e/2\gamma_\rho)(1-m_\rho^2/m_c^2). \tag{27}$$

This result coincides with the GMSW model[11] provided

$$\gamma_\rho(0) = (1-m_\rho^2/m_c^2)^{-1}\gamma_\rho. \tag{28}$$

Augustin et al.[12] have observed that the VMD result

$$\frac{\Gamma(\omega\to\pi\gamma)}{\Gamma(\omega\to3\pi)} = \text{const}\ \frac{\alpha}{\gamma_\rho(0)^2} \tag{29}$$

disagrees with experiment by a factor of 2 if they use their measured value for $\gamma_\rho(m_\rho^2)$ in place of $\gamma_\rho(0)$. By choosing $m_c^2 \simeq 2$, our correction factor $(1-m_\rho^2/m_c^2)^2$ in (28) puts (29) in perfect agreement with experiment. Augustin et al.[12] have also checked in VMD result

$$\frac{\Gamma(\omega\to\pi\gamma)}{\Gamma(\pi^0\to\gamma\gamma)} = \text{const}\ \frac{1}{\alpha}\frac{\gamma_\rho^2(0)}{4\pi}. \tag{30}$$

Taking $\Gamma(\pi^0\to\gamma\gamma) \simeq 7.5$ eV and using $\gamma_\rho(m_\rho^2)$, they found $\Gamma(\omega\to\pi\gamma) \simeq 410$ keV, in disagreement with the experimental value $\Gamma_{\exp}(\omega\to\pi\gamma) \simeq 1.1$ MeV. Using our result (28) with $m_c^2 \simeq 2$, we obtain $\Gamma(\omega\to\pi\gamma) \simeq 800$ keV in (30), which is quite an improvement. Using our modified VMD prediction (26) and the latest value[13] for $\Gamma(\pi^0\to\gamma\gamma)$, we obtain $\Gamma(\omega\to\pi\gamma) \simeq 1.2$ MeV, in good agreement with experiment.

We see that the effect of the continuum is to give a k^2 dependence to the γ–vector-meson junction which is quite appreciable. The striking fact is that with $m_c^2 \simeq 2\ \mathrm{BeV}^2$, one can account for the discrepancies of the VMD model discussed above. We are presently applying these same methods to study photoproduction.

We believe that the applications discussed in this note further demonstrate the usefulness of operator-product expansions near the light cone. Using essentially the same assumptions supported by the massive photon scattering experiments, we have attempted to justify and extend low-lying saturation of mass dispersion relations in both the pion and vector-meson channels. The light cone was thus seen to illuminate the nature of these old physical principles.

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†Alfred P. Sloan Foundation Fellow. Present address: New York University, 251 Mercer St., New York, N. Y. 10012.

‡Fulbright Fellow.

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[10]The light-cone contribution in Eq. (23) has been estimated in this way. Recalling (6) we have $F_1(1)\int_0^\infty e^{i\lambda}f(\lambda)d\lambda$. According to our previous assumption that $f(\lambda)$ is a function which is peaked around the origin we get as an *order-of-magnitude* estimate $F_1(1)\simeq f(0)$; taking the equal-time limit in (22) we get ($\mu=i$, $\nu=j$)

$$i\epsilon_{ijk}\,p_k f(0)=(2/\sqrt{3})e^2\int d^3x\,\langle 0|[J_i^{\,3}(x,0),J_j^{\,8}(0)]|p,\pi^0\rangle.$$

In the gluon model this commutator turns out to be

$$\int d^3x\,\langle 0|[J_i^{\,3}(\vec{x},0),J_j(0)]|\pi^0,p\rangle=i\epsilon_{ijk}3^{-1/2}\langle 0|A_k^{\,3}|\pi^0,p\rangle$$

which gives $f(0)\simeq(2e^2/3\sqrt{2})f_\pi$.

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Fluctuation of Cosmic-Ray Anisotropy and the Dimensionality of Propagation

Frank C. Jones

Theoretical Studies Branch, Goddard Space Flight Center, Greenbelt, Maryland 20771

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The relationship between the dimensionality of the cosmic-ray propagation function and the statistical distribution of anisotropies is demonstrated and an argument is presented in favor of "essentially" one-dimensional propagation. This implies that a fluctuation explanation of low observed anisotropy cannot be ruled out as has been stated by previous authors.

In a recent Letter Ramaty, Reames, and Lingenfelter[1] report results of a Monte Carlo calculation in which the injection of cosmic rays into the galaxy is considered to be a sequence of random discrete events in space-time. In their Letter they state that their results are in conflict with a suggestion[2] of the present author that small values of the cosmic-ray anisotropy could result from the statistical nature of the injection mechanism. This remark is based on the result[3] that in the distribution of anisotropies small values are suppressed and the maximum-likelihood value is of the order of the rms value. Ramaty, Reames, and Lingenfelter also state[1] that a fluctuation origin of small anisotropy is only possible in the case of "strictly one-dimensional" propagation of cosmic rays in the galaxy. This is demonstrated[3] by the result that when two of the dimensions were suppressed in the Monte Carlo calculation the suppression of small values was not observed and a distribution was obtained that was flat down to zero.

It is the purpose of this note to point out the reason for this dependence on dimensionality, to demonstrate that small values of anisotropy are not improbable if the propagation is "essentially" one dimensional, and to argue that this, in fact, is the case in our galaxy. In the following we shall consider the bulk cosmic-ray flux density $\vec{J}$ rather than the anisotropy δ since the former has simpler statistical properties and the latter quantity is simply related to it by $\delta\propto|\vec{J}|/\rho$,