Breakdown of the Pomeranchuk Theorem and the Behavior of the Leading J-Plane Singularity

Jiro Arafune and Hirotaka Sugawara
Institute for Nuclear Study, University of Tokyo, Tanashi, Tokyo, Japan
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We prove that the leading J-plane singularity in the symmetric partial-wave amplitude $F_J^+(t)$ near $t=0$ should behave like $\alpha_\pm(t)=1\pm At^{1/2}+$terms of higher order in t; namely, the Pomeranchuk pole (or cut) must be a pair of complex-conjugate poles (cuts) if the total cross sections $\sigma_T^{p,a}(s)\underset{s\to\infty}{\longrightarrow}$ const and $\sigma_T^p(\infty)\neq\sigma_T^a(\infty)$, where p and a denote particle-particle and antiparticle-particle scattering, respectively. We use only unitarity and analyticity to prove this.

Since the work of Pomeranchuk in 1958,[1] many theorists have tried to prove the so-called Pomeranchuk "theorem" starting from the axioms of relativistic quantum theory.[2] But it now seems almost clear that we need at least one ungrounded assumption (which essentially says ImF dominates ReF at high energy) to prove the "theorem." On the other hand, recent results from the Institute for High Energy Physics–CERN collaboration[3] show that $\sigma_T(K^-p)$ does not fall to $\sigma_T(K^+p)$ when the energy is supposedly high (55 GeV/c). If $\sigma_T(K^-p)-\sigma_T(K^+p)$ does not vanish in the high-energy limit, we will see a violation of the Pomeranchuk "theorem." Although the violation of the theorem may not be agreeable from the esthetic point of view,[4] we cannot exclude this case at the present time. If fact several authors have already proposed some models[5] motivated by the Serpukhov experiment.[3]

Our main object in this note is to derive some rigorous theoretical consequences of the breakdown of the Pomeranchuk "theorem." We also propose a model implied by this result. For simplicity we consider the scattering of a scalar particle (mass M) or antiparticle with a scalar target (mass M).

Now let us prove the following theorem:

Theorem 1.—If the particle cross section $\sigma_T^p(s)$ and the antiparticle cross section $\sigma_T^a(s)$ approach constants when $s\to\infty$ and $\sigma_T^p(\infty)\neq\sigma_T^a(\infty)$, then

$$\mathrm{Im}F^{p,a}(s,t)\underset{s\to\infty}{\geqslant}(K^{p,a}/16\pi)sJ_0(2\kappa^{p,a}\ln s(-t)^{1/2}),\tag{1}$$

when $0\leqslant t\leqslant t_0$ ($t_0=$threshold in the t channel). Here[6]

$$K^{p,a}=\frac{\{\pi^{-1}|\sigma_T^p(\infty)-\sigma_T^a(\infty)|-4\pi^{1/2}\kappa^{p,a}[\sigma_{el}^{p,a}(\infty)]^{1/2}\}}{C^2-(\kappa^{p,a})^2},\tag{2}$$

and $\kappa^{p,a}$ is a constant satisfying

$$0<\kappa^{p,a}\leqslant\frac{1}{4\pi^{3/2}}\frac{|\sigma_T^p(\infty)-\sigma_T^a(\infty)|}{[\sigma_{el}^{p,a}(\infty)]^{1/2}}.\tag{3}$$

C is also a constant satisfying

$$C>N/\sqrt{t_0},\tag{4}$$

where N is the number of subtractions in the Mandelstam representation. If we do not assume the Mandelstam representation but quantum field theory, N is the degree of the polynomial which gives the bound to the scattering amplitude.[7]

Proof.—We first expand $F^\pm(s,t)=F^p(s,t)\pm F^a(s,t)$ into partial waves:

$$F^{p,a}(s,t)=\frac{8\pi\sqrt{s}}{q_s}\sum_{l=0}^{\infty}(2l+1)a_l^{p,a}(s)P_l\left(1+\frac{2t}{s-4M^2}\right),\quad q_s\underset{s\to\infty}{\longrightarrow}\tfrac{1}{2}\sqrt{s}.\tag{5}$$

As was shown by Martin[7] we can neglect the summation above $l=Cx$, where $x=s^{1/2}\ln s$, if C satisfies the inequality (4). Schwartz's inequality gives us ($\kappa\equiv\kappa^{p,a}$)

$$16\pi\sum_{l=0}^{\kappa x}(2l+1)|\mathrm{Re}a_l^{p,a}(s)|\underset{s\to\infty}{\leqslant}\{[\sum_{l=0}^{\kappa x}(2l+1)|a_l^{p,a}(s)|^2][\sum_{l=0}^{\kappa x}(2l+1)]\}^{1/2}\underset{s\to\infty}{\leqslant}4\pi^{1/2}\kappa[\sigma_{el}^{p,a}(\infty)]^{1/2}s\ln s.\tag{6}$$

Since we have[1]

$$\lim_{s\to\infty} \mathrm{Re}F^-(s,0) \simeq 2\pi^{-1}[\sigma_T^{a}(\infty)-\sigma_T^{p}(\infty)]s\ln s, \quad \lim_{s\to\infty} \mathrm{Re}F^+(s,0)/s = 0, \tag{7}$$

and κ satisfies the inequality (3), we get

$$16\pi \sum_{l=\kappa x}^{Cx}(2l+1)|\mathrm{Re}a_l^{p,a}(s)| \underset{s\to\infty}{\geqslant} \{\pi^{-1}|\sigma_T^{a}(\infty)-\sigma_T^{p}(\infty)|-4\pi^{1/2}\kappa[\sigma_{el}^{p,a}(\infty)]^{1/2}\}s\ln s, \tag{8}$$

Because of unitarity we have

$$\mathrm{Im}F(s,t)^{p,a} \underset{s\to\infty}{\geqslant} 16\pi\sum_{\kappa x}^{Cx}(2l+1)\,\mathrm{Im}a^{p,a}(s)P_l\left(1+\frac{2t}{s-4M^2}\right) \underset{s\to\infty}{\geqslant} P_{\kappa x}\left(1+\frac{2t}{s-4M^2}\right)16\pi\sum_{\kappa x}^{Cx}(2l+1)|\mathrm{Re}a_l^{p,a}(s)|^2, \tag{9}$$

where $0 \leqslant t < t_0$. We have used the fact that $P_l(z)$ is an increasing function of l when $z \geqslant 1$. The summation in the right-hand side of (9) can be estimated in the following way:

$$\sum_{\kappa x}^{Cx}(2l+1)[(\mathrm{Re}a_l^{p,a}(s)]^2 \underset{s\to\infty}{\geqslant} [\sum_{\kappa x}^{Cx}(2l+1)|\mathrm{Re}a^{p,a}|]^2/[\sum_{\kappa x}^{Cx}(2l+1)]^{-1} \text{ (Schwartz inequality)}$$

$$\underset{s\to\infty}{\geqslant} \frac{1}{(16\pi)^2}\frac{\{\pi^{-1}|\sigma_T^{a}(\infty)-\sigma_T^{p}(\infty)|-4\pi^{1/2}\kappa[\sigma^{p,a}(\infty)]^{1/2}\}}{C^2-\kappa^2}s \tag{10}$$

[because of (8)]. Substituting (10) into (9) and using the appropriate asymptotic form for $P_l(z)$ we arrive at the theorem.

Next we proceed to the theorem concerning the upper bound:

Theorem 2.[8]—Under the same assumption as in the previous theorem we have

$$\mathrm{Im}F^{p,a}(s,t) \underset{s\to\infty}{\leqslant} \sigma_T^{p,a}(\infty)sJ_0(2C\ln s\sqrt{-t}), \tag{11}$$

when $0 \leqslant t \leqslant t_1 < t_0$.

Proof.—

$$\mathrm{Im}F^{p,a}(s,t) \underset{s\to\infty}{\sim} 16\pi\sum_{l=0}^{Cx}(2l+1)\,\mathrm{Im}a_l(s)P_l(1+2t/s) \underset{s\to\infty}{\lesssim} P_{Cx}(1+2t/s)\sum_{l=0}^{Cx}(2l+1)a_l(s)$$

$$\underset{s\to\infty}{\simeq} \sigma_T^{p,a}sP_{Cx}(1+2t/s).$$

Since $P_{Cx}(1+2t/s) \xrightarrow[s\to\infty]{} J_0(2C\ln s\sqrt{-t})$ we get the theorem.

As a consequence of these two theorems we get the following important corollary:

Corollary.—Under the same assumption as in Theorem 1, the leading J-plane singularity of $F_J^+(t)$ behaves like $\alpha_\pm(t) = 1 \pm A\sqrt{t}$ + terms of higher order in t near $t=0$, where A satisfies

$$2\kappa \leqslant A \leqslant 2C.$$

The proof is obvious if we use the asymptotic form of $J_0(z)$. This corollary does not tell us whether the pair of singularities is a pole or a cut but it tells that the singularities should collide at $t=0$.[9]

We can also show

$$\frac{K^{p,a}s}{16\pi}\frac{d^n}{dt^n}J_0(2\kappa t^{1/2}\ln s)\bigg|_{t=0} \leqslant \frac{d^n}{dt^n}\mathrm{Im}F^{p,a}(s,t)\bigg|_{t=0} \leqslant \sigma_T^{p,a}s\frac{d^n}{dt^n}J_0(2Ct^{1/2}\ln s)\bigg|_{t=0} \quad (n=0,1,2,\cdots)$$

using the same method as in the above theorems.

On the basis of above considerations we propose the following model for the high-energy scattering amplitude:

$$F^+(s,t) \underset{s\to\infty}{\sim} \sum_{i=+,-}\beta_i(t)\frac{1+e^{-i\pi\alpha_i(t)}}{\sin\pi\alpha_i(t)}s^{\alpha_i(t)}, \tag{12}$$

where

$$\alpha_\pm(t) = 1 \pm A\sqrt{t}, \quad \beta_+(t) = \beta_-(-\sqrt{t}),$$

and[10]

$$\mathrm{Im}F^{-}(s,t) \underset{s\to\infty}{\sim} \sigma_T{}^{p}(\infty)sJ_0(b^p \ln s\sqrt{-t}) - \sigma_T{}^{a}(\infty)sJ_0(b^a \ln s\sqrt{-t}). \tag{13}$$

Direct computation of $\mathrm{Re}F^{-}(s,t)$ using dispersion relation shows that we must have[11]

$$\sigma_T{}^{p}(\infty)/b^p = \sigma_T{}^{a}(\infty)/b^a \tag{14}$$

in order to get rid of the singularity at $t=0$. Then we get

$$\mathrm{Re}F^{-}(s,t) \underset{s\to\infty}{\sim} -\frac{s}{\pi^2}\frac{\sigma_T{}^{p}(\infty)}{b^p}\frac{1}{\sqrt{-t}}\int_{b^a(-t)^{1/2}\ln s}^{b^p(-t)^{1/2}\ln s} d\lambda J_0(\lambda), \tag{15}$$

When $b^{p,a}(-t)^{1/2}\ln s$ is large, the last integral is approximately

$$\sqrt{2}[C(y^p)-C(y^a)+S(y^p)-S(y^a)],$$

where

$$y^{p,a} = \{2b^{p,a}(-t)^{1/2}\ln s/\pi\}^{1/2}$$

and $C(y)$ and $S(y)$ are Fresnel's functions.

In conclusion we showed that when $\sigma_T{}^{p,a}(s) \underset{s\to\infty}{\longrightarrow}$ const and $\sigma_T{}^{p}(\infty) \neq \sigma_T{}^{a}(\infty)$ the leading singularity in $F_J{}^{+}(t)$ must be two complex-conjugate poles or cuts.[12] We think this result is remarkable since we have used only unitarity and analyticity to derive it. Our model for $F^{-}(s,t)$ is that of colliding cuts as in some of the papers of Ref. (5). It satisfies the scale invariance of Gribov et al.,[5] but because of possible oscillation of $F(s,x/\ln^2 s)$ when $s\to\infty$ we could not prove the validity of scale invariance in general.

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[1] I. Ya. Pomeranchuk, Zh. Eksp. Teor. Fiz. 7, 499 (1958) [Sov. Phys. JETP 34, 725 (1958)].

[2] For example, see R. J. Eden, Phys. Rev. Lett. 16, 39 (1966), and references quoted therein.

[3] J. V. Allaby *et al.*, Phys. Lett. 30B, 500 (1969).

[4] If the theorem does not hold in π-N scattering, the Adler-Weisberger sum rule breaks down, suggesting that we may have no commutation relations for change densities at all. Analysis of the forward dispersion relation and the Igi-Matsuda sum rule will also be incorrect.

[5] V. Barger and R. J. N. Phillips, Phys. Lett. 31B, 643 (1970); J. Finkelstein, Phys. Rev. Lett. 24, 172 (1970); V. N. Gribov, I. Yu. Kobsarev, V. D. Mur, L. B. Okun, and V. S. Popov, Phys. Lett. 32B, 129 (1970).

[6] We can show $\sigma_{el}{}^{p,a}(s) \underset{s\to\infty}{\gtrless}$ const. If it oscillates we must take the upper bound instead of $\sigma_{el}{}^{p,a}(\infty)$ in Eq. (2) and in the equations below.

[7] M. Froissart, Phys. Rev. 123, 1053 (1961); A. Martin, Nuovo Cimento 44, 1219 (1966); see also R. J. Eden, *High Energy Collisions of Elementary Particles* (Cambridge Univ., New York, 1967), p. 169.

[8] This is similar to the result of K. Bardakci except that ours is more restrictive since we assume $\sigma_T \to$ const. K. Bardakci, Phys. Rev. 127, 1832 (1962).

[9] This type of model was considered by some authors. For example, R. Oehme, Phys. Lett. 31B, 573 (1970).

[10] As was shown by Finkelstein (Ref. 5), a colliding-pole model is impossible for $F^{-}(s,t)$.

[11] This model for $F^{-}(s,t)$ corresponds to that of Gribov *et al.* (Ref. 5) with $xd^{-}(x) = (\sigma^a/b^a)[\theta(b^p-x)-\theta(b^a-x)]$ in their notation.

[12] Our proof is valid for $t \geq 0$. For $t<0$ we cannot exclude the possibility of, for example, the third singularity $[\alpha(t)=1+\alpha' t+\cdots]$ dominating the colliding singularities.