# Thermalization of a Magnetic Impurity in the Isotropic $XY$ Model

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(Received 18 September 1970)

We carry out an exact study of the relaxation of a single magnetic impurity embedded in a linear quantum chain. We obtain a rigorous result for the expectation value of any spin in the chain, and find that it relaxes to its equilibrium value as $t^{-1}$.

Recently, the time-dependent properties[1] of the $XY$ model[2] have been extensively investigated. For instance, it has been shown that if a uniform, time-dependent magnetic field is applied in the $z$ direction to <u>all the spins</u> of a system initially at equilibrium, then the average magnetization in the $z$ direction approaches a limit which is <u>not</u> its equilibrium value, no matter how slowly the field varies.[1,3] In this note, we obtain <u>rigorously</u> the magnetization at any site when a field which is applied to a single spin is removed. In the thermodynamic limit, we find that the magnetization of any <u>interior</u> spin approaches its new equilibrium value with time $t$ as $t^{-1}$. Tjon[4] has examined the behavior of a <u>boundary spin</u> in the weak-coupling approximation; approach to equilibrium again obtains, but as $t^{-3}$.

Consider the Hamiltonian

$$\mathcal{H}=\mathcal{H}+h(t)\sigma_{m,z} \tag{1}$$

where

$$\mathcal{H}_0=J\sum_{n=1}^{M}(\sigma_{n,x}\sigma_{n+1,x}+\sigma_{n,y}\sigma_{n+1,y}) \tag{2}$$

and

$$\begin{aligned} h(t)&=h, \quad t\leqslant 0,\\ &=0, \quad t>0. \end{aligned} \tag{3}$$

The $\sigma_\alpha$ are Pauli spin matrices, $J$ is the coupling constant, and $\sigma_{M+1,\alpha}=\sigma_{1,\alpha}$.

We assume that for $t\leqslant 0$ the system is in thermal equilibrium at temperature $\beta^{-1}$, and that at $t=0$ the magnetic field $h$ is switched off.

The magnetization of the $n$th spin is defined as

$$\langle\sigma_{n,z}(t)\rangle=\mathrm{Tr}[\exp(-\beta\mathcal{H})\exp(i\mathcal{H}_0 t)\sigma_{n,z}\exp(-i\mathcal{H}_0 t)]/\mathrm{Tr}\exp(-\beta\mathcal{H}) \tag{4}$$

and it thermalizes if

$$\lim_{t\to\infty}\langle\sigma_{n,z}(t)\rangle=0. \tag{5}$$

Since $H_0$ can be written in terms of fermion operators $a_q^\dagger$, $a_q$ as

$$\mathcal{H}_0=4J\textstyle\sum_q \cos q a_q{}^\dagger a_q \tag{6}$$

with

$$a_q{}^\dagger=M^{-1/2}\sum_1^M e^{iqm}(\sigma_{m,x}+i\sigma_{m,y})\prod_1^{m-1}\exp[i\pi(\sigma_{n,z}+1)/2],$$

we have, to within an irrelevant constant,

$$\mathcal{H}=4J\sum_q\cos q a_q{}^\dagger a_q+\frac{2h}{M}\sum_{qq'}e^{i(q'-q)m}a_q{}^\dagger a_{q'}. \tag{7}$$

Since $H$ is quadratic in $a_q$, $a_q{}^\dagger$, it can be written as

$$\mathcal{H}=\mathrm{const}+\textstyle\sum_q\lambda_j\alpha_j{}^\dagger\alpha_j, \tag{8}$$

where the $\alpha_j$ are new Fermi operators given by

$$\alpha_j = \sum_q U_{jq} a_q. \tag{9}$$

There are two kinds of eigenvalue $\lambda_j$:
(i) $\lambda_j \neq 4J\cos q$ for all $q$. Then

$$U_{jq} = e^{imq}[N(\lambda_j)(\lambda_j - 4J\cos q)], \tag{10}$$

where $N(\lambda_j)$ is the normalization factor and $\lambda_j$ are the zeros of

$$F(\lambda) = 1 - (2h/M)\sum_q(\lambda - 4J\cos q)^{-1}. \tag{11}$$

(ii) $\lambda_j = 4J\cos q_0$ for some $0 < q_0 < \pi$; then

$$U_{jq} = 2^{-1/2}(e^{imq_0}\delta_{qq_0} - e^{-imq_0}\delta_{q,-q_0}). \tag{12}$$

Combining these results, we obtain the final result in the thermodynamic limit:

$$\langle\sigma_{n,z}(t)\rangle = \frac{2h}{\pi i}\oint_C d\lambda \left\{(1+e^{\beta\lambda})^{-1}\left[1 - \frac{h}{\pi}\int_0^{2\pi}(\lambda - 4J\cos q)^{-1}dq\right]^{-1}\right.$$
$$\left.\times(2\pi)^{-2}\int_0^{2\pi}\int dq\, dq' \exp i[(m-n)(q-q') + 4J(\cos q - \cos q')t][(\lambda - 4J\cos q)(\lambda - 4J\cos q')]^{-1}\right\}, \tag{13}$$

where the contour $C$ is an ellipse which contains the zeros of $F(\lambda)$, but not those of $1+e^{\beta}\lambda$. By the Riemann-Lebesgue lemma, thermalization in the sense of (5) occurs. An asymptotic study of $\langle\sigma_{n,z}(t)\rangle$ from (13) gives a leading term proportional to $t^{-1}$, which is a rather slow approach to equilibrium. An interesting feature of (13) is its analytic properties for fixed $t$ around $h=0$.

The anisotropic case, as well as details of the present case, will be published elsewhere.

The authors wish to thank Professor J. Lebowitz for many interesting discussions. Two of us (D.B.A. and E.B.) wish to thank Professor Barry McCoy for a very fruitful discussion. D. B. Abraham wishes to thank Professor Mark Kac at the Rockefeller University for his kind hospitality.

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*Work supported in part by National Science Foundation Grant No. GP 9414.

†Part of this work was carried out at the IBM Watson Laboratory, Columbia University, New York, N. Y.

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