OFF-SHELL CONTINUATION OF THE TWO-NUCLEON TRANSITION MATRIX WITH A BOUND STATE

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(Received 21 May 1970)

A method of continuing the two-nucleon transition matrix off the energy shell for bound-state, partial-wave eigenchannels is presented. The arbitrary parmeters of this method are a symmetric function of momentum $\sigma_W(k,k')$ and the bound-state wave function $\Psi_B(r)$. The phase-shifts are essentially contained in $\sigma_W(k,k)$. While a potential is presumed to be present, it never explicitly appears in determining the T matrix.

Recently Baranger et al.[1] (BGMS) have demonstrated how the two-nucleon, nonrelativistic transition matrix may be continued off the energy shell without explicitly introducing a two-nucleon potential. The formalism of BGMS applies to partial-wave eigenchannels without a bound state. In this paper we extend the BGMS methods to eigenchannels with a bound state.

The two-nucleon transition matrix $T(\omega)$ is defined by the integral equation

$$\langle k'|T(\omega)|k\rangle = \langle k'|V|k\rangle + \int_0^\infty dq \langle k'|V|q\rangle(\omega-q^2)^{-1}\langle q|T(\omega)|k\rangle. \tag{1}$$

For $\omega = k^2+i\epsilon$, where ϵ is a positive infinitesimal, Eq. (1) may be derived from the Schroedinger equation for the outgoing scattering solution Ψ_k^+. In this case $\langle k'|T(k^2+i\epsilon)|k\rangle = \langle k'|V|\Psi_k^+\rangle$. The matrix elements $\langle k'|T(k^2+i\epsilon)|k\rangle$ are collectively referred to as the "half-on-shell" T matrix. Furthermore, we can define a real "half-shell" function $\varphi(k,k')$ by

$$\varphi(k,k') = \langle k'|V|\Psi_k^0\rangle, \tag{2}$$

where, for a given partial-wave eigenchannel, Ψ_k^0 is the real, delta-function normalized solution to the Schroedinger equation.

According to BGMS, one may construct the full T matrix given the symmetric part $\sigma(k,k')$ of the half-shell function $\varphi(k,k')$. Starting with an arbitrary $\sigma(k,k')$, one may follow the formalism and numerical methods of BGMS to obtain a T matrix that satisfies Eq. (1) for some unspecified short-range potential V. Since the diagonal elements of σ are directly related to the two-nucleon phase shifts, a correct fit to the experimental phase shifts is assured as long as $\sigma(k,k')$ has the appropriate values. Thus, the tedious process of phase-shift fitting is eliminated if one starts with σ instead of V to determine T. Furthermore, in many nuclear problems—for example, proton-proton bremsstrahlung,[2,3] the three-body problem,[4] and nuclear matter[5]—it is the T matrix, not the potential, that is closely related to the observables.

For partial-wave eigenchannels with a bound state, we will show that the off-shell continuation of the T matrix is determined by the bound-state wave function and a symmetric function of momentum. The diagonal elements of the symmetric function are directly related to the phase shifts. The only restriction we place on the transition matrix, as in BGMS, is that it is ultimately derivable from a nonrelativistic Schroedinger equation with a short-range, two-nucleon potential. In this way we guarantee that the scattering solutions and the bound-state solution form a complete orthonormal set.

We now briefly review the BGMS method. We will then show how it can be modified to accommodate the bound state.

For a partial-wave eigenchannel without a bound state the real scattering solutions $|\Psi_k^0\rangle$ form a complete, orthonormal set of wave functions (in the delta-function sense). This means that the unitary matrix U, which connects the unperturbed and scattering eigenstates, i.e.,

$$\langle k'|U|k\rangle = \langle k'|\Psi_k^0\rangle, \tag{3}$$

is real orthogonal. One may then express U as the sum of a symmetric part (S) and an antisymmetric part (A). The unitarity conditions then become

$$S^2-A^2=1, \quad AS-SA=0. \tag{4}$$

The matrix U may also be expressed in terms of the half-shell function $\varphi(k,k')$. The resulting expression for U, which can be derived from the Schroedinger equation, is

$$\langle k'|U|k\rangle = \cos\delta(k)\delta(k-k') + \mathrm{P}(\varphi(k,k')/(k^2-k'^2)), \tag{5}$$

where P stands for principal value. The two-nucleon phase shifts $\delta(k)$ are related to the diagonal values of φ by $\varphi(k,k) = -(2k/\pi)\sin\delta(k)$.

According to BGMS, the half-shell function may be expressed as the sum of a symmetric part $\sigma(k, k')$ and an antisymmetric part $\alpha(k,k')$. They then describe how to determine α, given σ, such that Eq. (4) holds. Since $\sigma(k,k) = \varphi(k,k)$, a fit to the experimental two-nucleon scattering data is fulfilled as long as the diagonal elements of σ yield the correct two-nucleon phase shifts.

If a bound state is present in the partial-wave eigenchannel, the real matrix U is not unitary. Orthonormality guarantees that $U^\dagger U = 1$, but lack of completeness of the scattering states means that $UU^\dagger \neq 1$. In this case Eq. (4) no longer holds. The numerical methods of BGMS, which rely on the validity of Eq. (4), are then inapplicable in determining α from σ.

We shall now show how to construct a real orthogonal matrix W for which Eq. (4) is valid. We deal with the operator W that connects the scattering states $|\chi_k^0\rangle$ of a model Hamiltonian, $H_M = K + V_M$, (K = kinetic energy) with the true scattering states $|\Psi_k^0\rangle$. The restriction we shall place on V_M is that it gives the same bound-state wave function and eigenvalue as the potential V, from which U and φ are derivable.

We define the real matrix W by

$$W(k',k) \equiv \langle\chi_{k'}^0|W|\chi_k^0\rangle = \langle\chi_{k'}^0|\Psi_k^0\rangle. \tag{6}$$

Because the bound-state wave functions $|\chi_B\rangle$ and $|\Psi_B\rangle$ are equal, it follows that

$$\langle\chi_k^0|\Psi_B\rangle = \langle\chi_k^0|\chi_B\rangle = 0; \quad \langle\Psi_k^0|\chi_B\rangle = \langle\Psi_k^0|\Psi_B\rangle = 0. \tag{7}$$

By applying the completeness of the eigenfunctions of H and H_M, and employing relation (7), one can easily show that

$$\int_0^\infty dq\, W(k',q)W(k,q) = \delta(k'-k), \quad \int_0^\infty dq\, W(q,k')W(q,k) = \delta(k'-k); \tag{8}$$

i.e., the matrix elements $W(k',k)$ form a real orthogonal matrix.

Since the matrix W is real orthogonal, we should presumably be able to parametrize it by a means similar to the BGMS prescription. To do this we define a new "half-shell" function $\varphi_W(k,k')$ by

$$\varphi_W(k,k') = \langle\chi_{k'}^0|V-V_M|\Psi_k^0\rangle. \tag{9}$$

We may then cast the well-known outgoing scattering equation for two-potential scattering[6] from a potential $V = V_M + (V-V_M)$,

$$|\Psi_k^+\rangle = |\chi_k^+\rangle + (k^2 - H_M + i\epsilon)^{-1}(V-V_M)|\Psi_k^+\rangle \tag{10}$$

in a form amenable to treatment with real wave functions. Employing the facts that $|\Psi_k^0\rangle = |\Psi_k^+\rangle e^{-i\delta(k)}$, and $|\chi_k^0\rangle = |\chi_k^+\rangle e^{-1\delta_M(k)}$, we obtain

$$|\chi_k^0\rangle = e^{i[\delta_M(k)-\delta(k)]}|\chi_k^0\rangle + (k^2-H_M+i\epsilon)^{-1}(V-V_M)|\Psi_k^0\rangle, \tag{11}$$

where δ_M is the phase shift for scattering from V_M alone.

By applying the definitions of W and φ_W, and the relation

$$(k^2-k'^2+i\epsilon)^{-1} = \mathrm{P}((k^2-k'^2)^{-1}) - i(\pi/2k)\delta(k-k'), \tag{12}$$

we obtain

$$W(k',k) = [e^{-i[\delta_M(k)-\delta(k)]} - i(\pi/2k)\varphi_W(k,k)]\delta(k-k') + \mathrm{P}(\varphi_W(k,k')/(k^2-k'^2)). \tag{13}$$

We may now express $\varphi_W(k,k)$ as a simple function of $\delta(k)$ and $\delta_M(k)$. To do this we expand the definition (9) in terms of a complete set of unperturbed intermediate eigenstates $|k\rangle$. We then use Eqs. (2) and (5), and the definition (3) of U to obtain

$$\varphi_W(k,k') = \cos\delta_M(k')\varphi(k,k') - \cos\delta(k)\varphi_M(k',k) + \mathrm{P}\int_0^\infty dq\, \varphi_M(k'q)\varphi(k,q)[(q^2-k^2)^{-1} - (q^2-k'^2)^{-1}]. \tag{14}$$

Here φ_M is the half-shell function for scattering from V_M alone.

For $k=k'$, the integral term in Eq. (14) vanishes and the diagonal elements of φ_W are given by

$$\varphi_W(k,k) = -(2k/\pi)\sin[\delta(k)-\delta_M(k)]. \tag{15}$$

Substitution of Eq. (15) into Eq. (13) yields

$$W(k',k) = \cos[\delta(k)-\delta_M(k)]\,\delta(k-k') + \mathrm{P}(\varphi_W(k,k')/(k^2-k'^2)). \tag{16}$$

Therefore, W is a real orthogonal matrix whose relation to the half-shell function φ_W is virtually the same as the relation (5) of U to φ. The only difference is that the phase-shift difference $\delta-\delta_M$ appears in Eq. (16), whereas the phase shift δ appears in (5). The half-shell function $\varphi_W(k,k')$ can be expressed as the sum of a symmetric function $\sigma_W(k,k')$ and an antisymmetric function $\alpha_W(k,k')$. The antisymmetric part may be determined from σ_W by the methods of BGMS. The only alteration to their method is that now the phase-shift difference $\delta-\delta_M$ plays the role previously played by the phase shift δ. The diagonal elements of σ_W are directly related to the phase-shift differences.

The next task is to relate the true half-shell function $\varphi(k,k')$ to $\varphi_W(k,k')$. By writing $V=V_M+(V-V_M)$, and expanding the definition (3) in terms of the complete set of eigenfunctions of H_M, we obtain

$$\begin{aligned}\varphi(k,k') = \cos[\delta(k)-\delta_M(k)]\varphi_M(k,k') + \cos\delta_M(k')\varphi_W(k,k') \\ + (k'^2-k^2)\mathrm{P}\int_0^\infty dq\, \varphi_M(q,k')\varphi_W(q,k)(q^2-k^2)^{-1}(q^2-k'^2)^{-1} \\ + \langle k'|V_M|\chi_B\rangle\langle\chi_B|\Psi_k^{\,0}\rangle + \langle k'|\chi_B\rangle\langle\chi_B|V-V_M|\Psi_k^{\,0}\rangle.\end{aligned} \tag{17}$$

The next-to-last term of Eq. (17) vanishes by Eq. (7). Since $|\chi_B\rangle$ is a bound-state eigenfunction of both H and H_M with the same eigenvalue, the last term in Eq. (17) also vanishes. Therefore, the half-shell function φ may be directly computed from the known φ_M, and from φ_W.

Once $\varphi(k,k')$ is known, we may compute the fully off-shell T matrix $T(\omega)$. We use the BGMS result, modified for the bound state, to obtain

$$\begin{aligned}\langle k'|T(\omega)|k\rangle = \varphi(k,k')\cos\delta(k) + \int_0^\infty dq\left(\frac{1}{\omega-q^2} - \frac{\mathrm{P}}{k^2-q^2}\right)\varphi(q,k')\varphi(q,k) \\ + \frac{(k_B^{\,2}+k'^2)(k^2-\omega)}{(\omega+k_B^{\,2})}\langle k'|\chi_B\rangle\langle\chi_B|k\rangle.\end{aligned} \tag{18}$$

Equations (17) and (18) indicate that an explicit knowledge of V is unnecessary in the evaluation of φ and $T(\omega)$. Starting with the symmetric function $\sigma_W(k,k')$, one may determine $T(\omega)$ without reference to the underlying potential V. The transition matrix $T(\omega)$ is a very convenient starting point in the study of inelastic processes, the three-body problem, and nuclear matter.

We have shown that it is possible to continue the two-nucleon transition matrix off the energy shell in a bound-state, partial-wave eigenchannel without direct reference to a two-nucleon potential. The parameters of our formalism are essentially the symmetric function $\sigma_W(k,k')$ and the model potential V_M. If one wishes, one may employ the bound-state wave function instead of V_M since for certain types of model potentials (e.g., separable potentials) the model potential can be calculated directly from $\chi_B(r)$. By employing this procedure, one can fix certain bound-state observables as well as the phase-shifts [which are essentially contained in $\sigma_W(k,k)$]. One may then test the sensitivity of certain nuclear processes to off-energy-shell properties of the T matrix and to the bound-state (deuteron) wave function.

We are currently testing the formalism described above with several solvable models.

The author would like to thank Dr. S. K. Mukhopadhyay and Dr. P. U. Sauer for helpful conversations.

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