that deformation is expected to occur abruptly between 86 and 88 neutrons for the nuclei discussed here. These calculations, which are based on the Nilsson model, combined with the Strutinsky normalization procedure, reproduce the general trend of decreased deformation for nuclei with 88 neutrons on both sides of $Z = 62$.

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EQUATION OF STATE FOR DENSE NEUTRON MATTER

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A solid-body model for neutron matter is proposed for calculating the equation of state for densities beyond the nuclear density using the Reid soft-core potential for the neutron-neutron interaction.

The equation of state for cold, catalyzed matter has been studied by a number of workers[1-9] for densities of the order of nuclear density ($\rho \sim 3\times10^{14}$ g/cm^3). However the extension of the existing calculations to higher densities is beset with two difficulties: (i) an inadequate knowledge of the nucleon-nucleon interaction at high energies and (ii) the lack of reliable methods for computing the interaction energy of the system at densities higher than the nuclear density. Even if we ignore the first difficulty, the second one is in itself formidable. The methods used in the works cited, generally, are designed for densities near nuclear density and do not work well at higher densities. For example, Cameron and his collaborators[3,4] have studied extensively the properties of neutron stars over a wide range of densities using the velocity-dependent potential of Levinger and Simmons,[10] where the potential energy is calculated only in the first-order perturbation theory. Recently Nemeth and Sprung[7] have calculated the energy of a system of neutrons and protons using the more realistic Reid[11] potential and the reaction-matrix theory of Brueckner and Goldstone. Similar calculations for a system of neutrons have been done by Binder, Pierce, and Razavy[8] and by Wang, Rose, and Schlenker,[9] using different potentials. These calculations are only carried out in the first order of the Brueckner-Goldstone theory; at higher densities than nuclear, higher orders of that theory become important and are almost prohibitively difficult to calculate. It is, therefore, desirable to have a better knowledge of the equation of state at high densities ($\rho \geqslant 10^{15}$ g/cm^3) to construct more realistic neutron-star models, and even apart from studying the structure of neutron stars, the problem has an intrinsic importance from the point of view of understanding the behavior of highly condensed matter.

In this paper we wish to explore the possibility of extending the equation of state beyond the nuclear density region. For this purpose we purpose a solid-body model for dense neutron matter[12] (for simplicity we ignore the presence of other particles) at zero temperature. The neutrons are assumed to form a body-centered cubic lattice, with lattice distance a such that the neutron at the center of the cell has a spin opposite to that of the eight neutrons at the corners (such an arrangement would give the most attraction). In the lattice calculations we take into account interactions up to the fourth-nearest neighbors. The interaction between the neutrons with opposite spins is assumed to be

$$v_1(r)=\tfrac{1}{2}(V_{\text{singlet}}+V_{\text{triplet}}),$$

where the singlet potential is given by the ${}^1S_0$ Reid[11] potential,

$$V_{\text{singlet}}(r)=(\mu r)^{-1}(Ae^{-\mu r}+Be^{-4\mu r}+Ce^{-7\mu r}). \qquad (1)$$

The constants A, B, and C (in MeV) are, respectively, -10.463, -1650.6, and 6484.2 and $\mu=0.7$ fm^{-1}. For the triplet potential we make the reasonable assumption that it is purely repulsive and that it is identical with the repulsive part of $V_{\text{singlet}}(r)$, i.e.,

$$v_2(r)=(\mu r)^{-1}Ce^{-7\mu r}. \qquad (2)$$

The vibrational energy of the solid can be calculated in the harmonic approximation using standard methods.[13] Let N be the number of lattice points in each direction so that there are a total of $2N^3$ neutrons. Assuming that the displacements from the equilibrium position of the lattice point are small, the potential energy may be expanded in Taylor's series,

$$\sum_{i<j=1}^{2N^3} v(\vec{r}_i-\vec{r}_j)=\sum_{i<j=1}^{2N^3} v(a_{ij})+\frac{1}{2}\sum_{i<j=1}^{2N^3}(\vec{\xi}_i-\vec{\xi}_j)^2 v_{ij}''(a_{ij}). \qquad (3)$$

Here $\vec{\xi}_i$ is the displacement vector of the ith neutron from its equilibrium position and $v_{ij}''(a_{ij})$ is the second derivative of $v(r)$ evaluated at the equilibrium position. Then the secular equation for the oscillation angular frequency of the lattice is obtained in the standard way. It is a 6×6 determinant which factorizes into two 3×3 determinants,

$$\begin{vmatrix} d_1\mp b_1-m\omega^2 & b_{12}\pm a_{12} & b_{13}\pm a_{13}\\ b_{12}\pm a_{12} & d_2\mp b_2-m\omega^2 & b_{23}\pm a_{23}\\ b_{13}\pm a_{13} & b_{23}\pm a_{23} & d_3\mp b_3-m\omega^2\end{vmatrix}=0, \qquad (4)$$

where

$$\begin{aligned} d_k&=2\alpha+4\beta\sin^2\varphi_k+4\gamma-2\gamma\cos2\varphi_k(\cos2\varphi_i+\cos2\varphi_j)+(22/3)\delta,\\ b_k&=2\alpha\cos\varphi_1\cos\varphi_2\cos\varphi_3+\frac{\delta}{3}\sum_{l,m,n}\cos3\varphi_l\cos\varphi_m\cos\varphi_n+\frac{16}{3}\delta\cos3\varphi_k\cos\varphi_i\cos\varphi_j,\\ b_{ij}&=2\gamma\sin2\varphi_i\sin2\varphi_j,\\ a_{ij}&=2\alpha\sin\varphi_i\sin\varphi_j\cos\varphi_k+2\delta\sin3\varphi_i\sin\varphi_j\cos\varphi_k+2\delta\sin\varphi_i\sin3\varphi_j\cos\varphi_k\\ &\quad+\tfrac{2}{3}\delta\sin\varphi_i\sin\varphi_j\cos3\varphi_k, \end{aligned} \qquad (5)$$

with

$$i\neq j\neq k=1,2,3, \text{ and } l\neq m\neq n=1,2,3.$$

Here

$$\alpha=\tfrac{4}{3}v_1''\left(\frac{\sqrt{3}}{2}a\right),\quad \beta=v_2''(a),$$

$$\gamma=v_2''(\sqrt{2}a),\quad \delta=\frac{12}{11}v_1''\left(\frac{\sqrt{11}}{2}a\right), \qquad (6)$$

and

$$\varphi_i=2\pi C_i/N, \qquad (7)$$

where C_i are integers such that $-\frac{1}{2}N\leq C_i\leq\frac{1}{2}N$.

Further, m is the neutron mass and $\frac{1}{2}\sqrt{3}a$, a, $\sqrt{2}a$, and $\frac{1}{2}(11)^{1/2}a$ are, respectively, first-, second-, third-, and fourth-nearest-neighbor distances. The maximum (Debye) frequency given by the combination $(0, \frac{1}{2}\pi, \frac{1}{2}\pi)$ for $(\varphi_1, \varphi_2, \varphi_3)$ is

$$\omega_D^2=(4/m)(\alpha+\beta+\gamma+\delta). \qquad (8)$$

We assume for the frequency spectrum the Debye expression

$$g(\omega)=C\omega^2,$$

where C is determined by the condition that the total number of modes be equal to $6N^3$,

$$\int_0^{\omega_D} g(\omega)\, d\omega = 6N^3,$$

which gives

$$C = 18N^3/\omega_D^3. \tag{9}$$

The vibrational energy density is then

$$\epsilon_{\rm vib} = (N^3a^3)^{-1}\int_0^{\omega_D}\tfrac{1}{2}\hbar\omega\, g(\omega)\, d\omega,$$

$$= \frac{9}{4}\frac{\hbar}{a^3}\omega_D. \tag{10}$$

The total energy density is

$$\epsilon = \epsilon_{\rm rest} + \epsilon_{\rm static} + \epsilon_{\rm vib}, \tag{11}$$

where

$$\epsilon_{\rm rest} = (2/a^3)mc^2 \tag{12}$$

and

$$\epsilon_{\rm static} = a^{-3}[8v_1(\tfrac{1}{2}\sqrt{3}\,a) + 6v_2(a) + 12v_2(\sqrt{2}a) + 24v_1(\tfrac{1}{2}(11)^{1/2}a)]. \tag{13}$$

The pressure p is calculated from the relation

$$p = -\epsilon - \tfrac{1}{3}a\,d\epsilon/da. \tag{14}$$

The results of the numerical evaluation of Eqs. (10)-(14) are displayed in Table I for various values of the density $\rho = 2m/a^3$, and the pressure–energy-density plot is exhibited in Fig. 1.

It may be noticed from Table I that as ρ increases the sound speed given by $c_s = c(dp/d\epsilon)^{1/2}$ approaches the velocity of light c. Evidently our nonrelativistic treatment is no longer applicable and the oscillations of the neutron should be treated relativistically. In the absence of such a theory we have adopted the following procedure:

Table I. The total energy density ϵ and the total pressure p for various values for the density ρ. The values of q, the adiabatic index $\Gamma = [(p+\epsilon)/p]\,dp/d\epsilon$, and $dp/d\epsilon$ are also tabulated. Note that in the second, third, and fourth columns the digits after the plus sign refer to an exponent of 10; thus, for example, 7.65+14 stands for 7.65×10^{14}.

$x = \mu a$	ρ g/cm^3	ϵ erg/cm^3	p dyne/cm^2	q	$dp/d\epsilon$	Γ
1.145	7.65 + 14	7.28 + 35	2.88 + 35	0.476	0.147	0.52
1.120	8.18 + 14	7.99 + 35	3.08 + 35	0.431	0.377	1.35
1.090	8.87 + 14	8.94 + 35	3.52 + 35	0.393	0.515	1.83
1.060	9.65 + 14	1.01 + 36	4.14 + 35	0.365	0.602	2.06
1.030	1.05 + 15	1.14 + 36	4.98 + 35	0.342	0.668	2.19
1.000	1.15 + 15	1.29 + 36	6.07 + 35	0.323	0.724	2.27
0.970	1.26 + 15	1.48 + 36	7.48 + 35	0.307	0.775	2.31
0.940	1.38 + 15	1.71 + 36	9.31 + 35	0.293	0.823	2.33
0.910	1.53 + 15	1.99 + 36	1.17 + 36	0.280	0.868	2.35
0.880	1.69 + 15	2.34 + 36	1.48 + 36	0.269	0.911	2.35
0.845	1.90 + 15	2.87 + 36	1.97 + 36	0.258	0.958	2.35
0.810	2.16 + 15	3.56 + 36	2.66 + 36	0.248	1.00	2.35
0.576	6.01 + 15	2.45 + 37	2.36 + 37	-	-	2.04
0.486	1.00 + 16	6.71 + 37	6.62 + 37	-	-	2.01
0.284	5.02 + 16	1.675 + 39	1.674 + 39	-	-	2.001
0.225	1.01 + 17	6.771 + 39	6.770 + 39	-	-	~ 2.00
0.104	1.02 + 18	6.942 + 41	6.942 + 41	-	-	~ 2.00

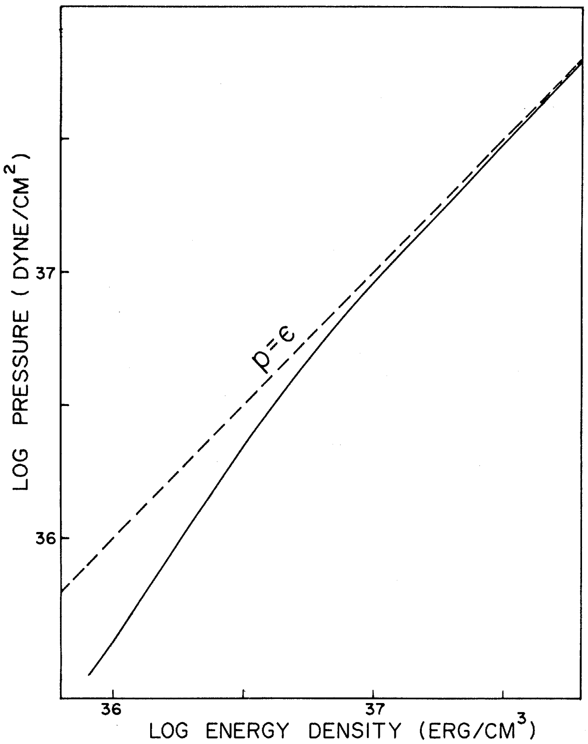

FIG. 1. Log pressure versus log energy density.

When c_s comes close to c at $\rho \sim 2.2\times 10^{15}\ g/\mathrm{cm}^3$, we set $dp/d\epsilon = 1$ and thereafter compute p and ϵ with the help of Eq. (14),

$$d\epsilon/da = -(3/a)(\epsilon + p).$$

In the high-density limit then p tends to ϵ in agreement with Zel'dovich's[14] result for a purely repulsive Yukawa potential. For low densities we cannot reliably extend our calculations below $\rho \sim 7.5\times 10^{14}$ g/cm^3 since the lattice structure becomes unstable as a result of the potential approaching its point of inflexion.

The calculation of the vibrational energy has been done in the harmonic approximation for the density range $7.5\times 10^{14} \leq \rho \leq 2.2\times 10^{15}$ g/cm^3. The physical requirement for the validity of the approximation that

$$q = \frac{(\hbar/\omega_D m)^{1/2}}{(\sqrt{3}/2)a},$$

the ratio of the amplitude of the zero-point oscillation to the nearest neighbor distance, be much smaller than unity is satisfied throughout this range. For $\rho \geq 2.2\times 10^{15}$, as pointed out earlier, the vibrational energy should be calculated relativistically. We have, however, adopted a simplified procedure for the purpose of the present calculation.

The proposed model therefore offers a possibility of extending the equation of state beyond the nuclear density. Admittedly, our model is somewhat idealized from the point of view of calculating the structure of highly condensed neutron stars in that it neglects the presence of protons, electrons, and other particles. Nevertheless, at least for neutron matter, our calculations indicate the nature of the equation of state at high densities.

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