upper bounds of T_h, for our sample of NaF, are ~5 and 10°K, respectively, for an ambient temperature of 2°K.

The theory predicts that the typical length (or time) for the development of the instability varies as σ^{-2}, where σ is the input width. In Fig. 3, we have plotted the output height H against σ^2 and find that for small σ, $H \sim \exp(\sigma^2)$ verifying that the time constant for the instability does indeed follow the predicted behavior.

In addition to the experiments described above, measurements were made in other heater-detector geometries. In one such experiment, a large heater (1.2×1.2 cm^2) was deposited on one end and a linear array of three detectors of size 3.5×3.5 mm^2 was deposited at the other end. It was found that each detector received approximately the same amount of energy at all power levels. Thus power-dependent spatial focusing effects are unimportant in our experiment. In another experiment heat pulses were propagated approximately 20° from the (100) direction. No pulse steepening was observed, confirming that propagation along a symmetry direction is necessary to observe the effects discussed here.

In summary, we have observed nonlinear propagation of heat pulses in solids in excellent agreement with the theory of Tappert and Varma.

We would like to thank F. D. Tappert and N. J. Zabusky for stimulating discussions, S. Geschwind for comments on the manuscript, and M. A. Chin for skillful technical assistance. We would also like to thank R. O. Pohl and D. J. Channin for supplying us the excellent NaF sample.

[1]F. D. Tappert and C. M. Varma, following Letter [Phys. Rev. Lett. 25, 1108 (1970)].

[2]For a recent review, see R. J. von Gutfeld, in *Physical Acoustics*, edited by W. P. Mason (Academic, New York, 1968), Vol. 5, p. 233.

[3]For simplicity we are neglecting acoustic-mismatch effects. Recent experiments show that for many solids this is approximately justified. See, for example, V. Narayanamurti, Phys. Lett. 30A, 521 (1969), and to be published.

[4]R. Orbach and L. A. Vredevoe, Physics (Long Is. City) 1, 91 (1964).

[5]By a suitable choice of biasing current and magnetic field our 1000-Å thick, slowly evaporated indium bolometers could be operated over a nearly linear transition region greater than 0.5°K in width without saturation at the high powers.

[6]T. F. McNelly, S. J. Rogers, D. J. Channin, R. J. Rollefson, W. M. Goubau, G. E. Schmidt, J. A. Krumhansl, and R. O. Pohl, Phys. Rev. Lett. 24, 100 (1970).

ASYMPTOTIC THEORY OF SELF-TRAPPING OF HEAT PULSES IN SOLIDS

F. D. Tappert
Bell Telephone Laboratories, Whippany, New Jersey 07981

and

C. M. Varma
Bell Telephone Laboratories, Murray Hill, New Jersey 07974

(Received 10 August 1970)

Using an asymptotic analysis leading to a modified Korteweg-de Vries equation and a nonlinear parabolic equation, it is shown that under certain conditions on phonon dispersion and lattice anharmonicity, self-trapping of heat pulses occurs in solids at low temperatures.

In an insulating crystal at an ambient temperature T_a in the ballistic regime of the thermal conductivity curve, a heat pulse generated at temperature $T_h > T_a$ may be regarded as a collection of large-amplitude waves propagating on a nonlinear lattice. The propagation of heat pulses is determined by the equations of motion,

$$m\ddot{u}_i^{\alpha} = \sum_{j\beta} a_{ij}^{\alpha\beta} u_j^{\beta} + \sum_{j\beta}\sum_{l\nu} a_{ijl}^{\alpha\beta\nu} u_j^{\beta} u_l^{\nu} + \sum_{j\beta}\sum_{l\nu}\sum_{m\delta} a_{ijlm}^{\alpha\beta\nu\delta} u_j^{\beta} u_l^{\nu} u_m^{\delta}, \quad (1)$$

where u_i^{α} is the displacement of the ith atom in the α direction, and the a's are derivatives of the interatomic potentials. Along a symmetry direction the propagation of different polarizations decouple ($a_{ij}^{\alpha\beta} \sim \delta_{\alpha\beta}$) in the harmonic approximation. Nonlinear terms with $\beta, \nu, \cdots = \alpha$ produce self-action effects, whereas nonlinear terms with $\beta, \nu, \cdots \neq \alpha$ produce coupling to other polarizations and therefore loss (or gain) to the pulse. Neglecting the short initial

time interval when the pulses having different polarizations overlap, the coupling of the lowest transverse branch (lt) to the other branches is very weak if $T_h \gg T_a$ since phonons in the lt branch can only decay by first combining with a phonon of one of the upper branches. Orbach and Vredevoe[1] have shown that for phonons of frequency ω such that $\hbar\omega/k_B \sim T_h \gg T_a$ (k_B = Boltzmann's constant), the linear relaxation times are given by

$$\tau_N^{-1} \sim T_h^5 \exp(-T_h/T_a) \text{ for the } lt \text{ branch,}$$
$$\sim T_h^5 \text{ for other branches.} \quad (2)$$

Therefore when $T_h \gg T_a$ and for propagation along a symmetry direction, we may neglect losses of the lt branch and use a scalar amplitude.

Neglecting diffraction effects, the wave motion is one-dimensional. We consider T_h such that the lt phonon dispersion is adequately represented by

$$\omega^2 = c^2k^2[1-h^2k^2+O(h^4k^4)],$$

where c is the lt sound velocity and h is a length on the order of a lattice spacing. Introducing the continuous Langrangian displacement $u(x,t)$, Eq. (1) reduces under the above conditions to the following nonlinear, dispersive, scalar wave equation[2]:

$$u_{tt} = c^2[u_{xx}(1+\epsilon p u_x + q u_x^2) + h^2 u_{xxxx}], \quad (3)$$

where the subscripts denote partial derivatives, ϵ is a small numerical constant, and p and q are numerical constants of order unity which are directly related to the anharmonic force constants: $\epsilon p = a_3 h/a_2$ and $q = a_4 h^2/a_2$, where a_2, a_3, and a_4 are essentially the second, third, and fourth derivatives of the potential. We have assumed that the quadratic nonlinearity (cubic term in the potential) is small compared with the cubic nonlinearity[3] (quartic term in the potential); the degree of smallness is measured by ϵ.

In order to describe only waves traveling toward the right, Eq. (3) is expanded along the characteristic $x-ct$ and at the same time made dimensionless and properly scaled [so that the nonlinear and dispersive terms in Eq. (3) are small, as required by its derivation]. To do this, let

$$\xi = \epsilon(x-ct)/h, \quad \tau = \epsilon^3 ct/h, \quad u/h = av, \quad (4)$$

where $a \ll 1$ is a parameter which measures the amplitude and $\xi = O(1)$, $\tau = O(1)$, $v = O(1)$. This ordering differs from that which leads to the Korteweg–de Vries (KdV) equation[4] because we are here interested in relatively high-frequency solutions of (3), where the dispersion is strong enough to prevent the rapid harmonic generation and wave-form distortion which occurs at lower frequencies because of resonant energy transfer between Fourier components.[5] Using (4) in (3) and setting $v_\xi = w$, we obtain the modified KdV equation,[6]

$$w_\tau + \tfrac{1}{2}(paw + qa^2w^2)w + \tfrac{1}{2}w_{\xi\xi\xi} = O(\epsilon^2), \quad (5)$$

where it is assumed that $\epsilon^2 \ll a^2$.

Next following the method outlined by Benney, Newell, and Roskes,[7] Karpman and Krushkal,[8] and Asano, Taniuti, and Yajima,[9] we perform an additional asymptotic expansion to obtain a representation of Eq. (5) corresponding to narrow-band wave packets. Thus we look for a solution of the form,

$$w = \psi^{(1)}e^{i(k_0\xi-\omega_0\tau)} + \text{c.c.} + a[\psi^{(0)} + \psi^{(2)}e^{2i(k_0\xi-\omega_0\tau)}$$
$$+ \text{c.c.}] + a^2[\psi^{(3)}e^{3i(k_0\xi-\omega_0\tau)} + \text{c.c.}] + O(a^3), \quad (6)$$

where k_0 is a positive parameter of order unity, and the $\psi^{(i)}$ are complex amplitudes of order unity which vary slowly with ξ and τ. Carrying out the expansion to $O(a^2)$ inclusive, we find that $\psi^{(1)}$ satisfies the nonlinear parabolic equation[7-9]

$$i\psi_s^{(1)} = \tfrac{3}{2}k_0\psi_{yy}^{(1)} + \tfrac{1}{2}k_0(q-p^2/6k_0^2)|\psi^{(1)}|^2\psi^{(1)}, \quad (7)$$

where $y = a(\xi + \tfrac{3}{2}k_0^2\tau)$ and $s = a^2\tau$. All quantities in Eq. (7) are now of order unity. Although this derivation, strictly speaking, applies only to coherent wave packets, it is clear from the form of the nonlinear term in Eq. (7) (no phase correlation) that incoherent wave packets (such as heat pulses in solids) may also be described by this equation if k_0 is chosen as a representative wave number of the "carriers" in the pulse.

From the known properties[8-11] of Eq. (7) it follows that finite-amplitude periodic waves are unstable, and self-trapping (more precisely, longitudinal pace-time self-focusing[10]) is possible if

$$h^2k_0^2 > \epsilon^2p^2/6q > 0, \quad (8)$$

where we have now reverted to dimensional units. Thus the quadratic nonlinearity has a stabilizing

effect which can be overcome only at sufficiently high frequencies. This instability does not appear in the KdV approximation ($q = 0$). Furthermore, in order that a pulse of width σ undergo self-trapping, its amplitude [$\langle u_x^2 \rangle = 2\epsilon^2 a^2 |\psi^{(1)}|^2$] must exceed a critical threshold,[11,12] i.e.,

$$\langle u_x^2 \rangle > \langle u_x^2 \rangle_c = \alpha \left(\frac{h}{\sigma}\right)^2 \left(q - \frac{\epsilon^2 p^2}{6h^2 k_0^2}\right)^{-1}, \tag{9}$$

where α is a numerical constant of order unity which depends on the pulse shape.

When condition (8) is satisfied, it is known[8,10] that Eq. (7) has a stationary solution wherein the linear dispersive spreading is exactly balanced by the nonlinear self-trapping. Such a wave packet, which has been called an envelope soliton[8] (for brevity, an E soliton), is distinct from the classical soliton[13] (solitary wave[14]) which can be obtained as a solution of Eq. (5) in a different asymptotic limit from that considered here. Expressed in terms of the original variables, the E soliton is given by

$$\langle u_x^2 \rangle = A^2 \operatorname{sech}^2[A(Q/24)^{1/2}(x - v_g t)/h], \tag{10}$$

where $A = 2\epsilon a$, $Q = q - \epsilon^2 p^2/6h^2 k_0^2$, and $v_g = c(1 - \frac{3}{2}h^2 k_0^2)$.

We now discuss the conditions on T_h to observe E solitons. We must, of course, satisfy (8); further, when $\langle u_x^2 \rangle \gg \langle u_x^2 \rangle_c$, the propagation distance required to achieve maximal self-trapping is given roughly by

$$d \sim \frac{\sigma^2}{h^2 k_0} \left(\frac{\langle u_x^2 \rangle c}{\langle u_x^2 \rangle}\right)^{1/2} \tag{12}$$

In order to observe the effect, we must have $d < L$, the crystal size. Neglecting zero-point motion, this yields a condition on T_h,

$$T_h^2 > T_{h(\text{lower})}^2 \simeq q^{-1/2} \frac{c\sigma \Theta_D \hbar}{L k_B} \left(\frac{m k_B \Theta_D}{\hbar^2}\right)^{1/2}, \tag{13}$$

where Θ_D is the Debye theta of the crystal and $\hbar$ and k_B are the Planck and Boltzmann constants, respectively. The upper limit on T_h is set by losses: that is, $d < \lambda$, the mean free path, which for Rayleigh scattering by defects is given by

$$\lambda \simeq \frac{4\pi\rho c^4}{r^2 x(1-x) k_B^4 T_h^4} \equiv \frac{\beta}{T_h^4},$$

where x is the concentration of the defects, r is the relative magnitude of the defect (for example, $r = \Delta m/m$ for defects of mass $m \pm \Delta m$), and ρ is the density of the crystal. We get the upper limit on T_h:

$$T_h^2 < T_{h(\text{upper})}^2 = (\beta/L) T_{h(\text{lower})}^{-2} \tag{14}$$

Next we examine the possibility of observing self-trapping of heat pulses in different T_a regimes.

(1) T_a in ballistic regime.—As already argued, for T_h satisfying Eqs. (12) and (14), E solitons should be observed.

(2) T_a in the second-sound regime.— The second-sound regime is defined by[15]

$$\tau_N^{-1}(\omega, T_a) \gg \sigma^{-1} c \gg \tau_u^{-1}(\omega, T_a) \tag{15}$$

(besides the condition on the impurity scattering time), where τ_N and τ_u are the normal and umklapp relaxation times. Normally second-sound experiments are done with heater temperatures very slightly above T_a, so that τ_N and τ_u are functions only of T_a. In the more general case τ_N and τ_u will be a function of the phonon frequencies ω in the heat pulse. If propagation is along a symmetry direction, second sound is observed[16] in the transverse branch since for $T_h \simeq T_a$, τ_N for transverse phonons is less than for longitudinal phonons. If now $T_h \gg T_a$, by (2) τ_N for transverse phonons will increase sharply and for longitudinal phonons will decrease. Thus the possibility exists in principle that for $T_h \gg T_a$, second sound may disappear for the transverse phonons and appear for the longitudinal phonons and E solitons may appear for the transverse phonons. This requires a simultaneous satisfaction of conditions (12), (14), and (15) appropriately for the transverse and longitudinal branches, and detailed numerical work is required to assess its feasibility in any given material.

(3) T_a in umklapp regime.— Equation (5) conserves the total momentum. Hence we do not believe that solitary waves can be seen for T_a in the umklapp regime for any T_h.

Since one does not obtain a one-dimensional equation of motion in a scalar variable for an arbitrary direction of propagation, we suspect that the nonlinear behavior discussed may not be present for such a propagation. The experimental results reported by Narayanamurti and Varma[17] for T_a in ballistic regime are fully consistent with the predictions of the theory presented above.

We appreciate many helpful remarks by G. Roskes who has independently verified the asympotic analysis presented here. We wish to thank V. Narayanamurti and N. J. Zabusky for stimulating discussions.

[1]R. Orbach and L. A. Vredevoe, Physics (Long Is. City) 1, 91 (1964).

[2]N. J. Zabusky, in *Nonlinear Partial Differential Equations*, edited by W. Ames (Academic, New York, 1967); N. J. Zabusky, J. Phys. Soc. Jap., Suppl. 26, 196 (1969).

[3]In cubic crystals, nearest-neighbor force constants of the type a_3 are zero on symmetry grounds.

[4]C. S. Gardner and G. K. Morikawa, New York University Institute of Mathematical Sciences Report No. 9082, 1960 (unpublished); C. H. Su and C. S. Gardner, J. Math. Phys. (N. Y.) 10, 536 (1969).

[5]R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory* (Benjamin, New York, 1969).

[6]R. M. Muira, J. Math. Phys. (N. Y.) 9, 1202 (1968); R. M. Muira, C. S. Gardner, and M. D. Kruskal, J. Math. Phys. (N. Y.) 9, 1204 (1968).

[7]D. J. Benney, J. Math. Phys. (Cambridge, Mass.) 46, 115 (1967); D. J. Benney and A. C. Newell, J. Math. Phys. (Cambridge, Mass.) 46, 133 (1967); D. J. Benney and G. J. Roskes, Stud. Appl. Math. 48, 377 (1969).

[8]V. I. Karpman, Pis'ma Zh. Eksp. Teor. Fiz. 6, 829 (1967) [JETP Lett. 6, 277 (1967)]; V. I. Karpman and E. M. Krushkal, Zh. Eksp. Teor. Fiz. 55, 530 (1968) [Sov. Phys. JETP 28, 277 (1969)].

[9]T. Taniuti and N. Yajima, J. Math. Phys. (N. Y.) 10, 1369 (1969); N. Asano, T. Taniuti, and N. Yajima, J. Math. Phys. (N. Y.) 10, 2020 (1969).

[10]L. A. Ostrovskii, Zh. Eksp. Theor. Fiz. 51, 1189 (1966) [Sov. Phys. JETP 24, 797 (1967)].

[11]S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Usp. Fiz. Nauk 93, 19 (1967) [Sov. Phys. Usp. 10, 609 (1968)].

[12]P. L. Kelley, Phys. Rev. Lett. 15, 1005 (1965).

[13]N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. 15, 240 (1965).

[14]D. J. Korteweg and G. de Vries, Phil. Mag. 39, 422 (1895).

[15]See for example, C. C. Ackerman and R. A. Guyer, Ann. Phys. (New York) 50, 128 (1968).

[16]T. F. McNelly *et al.*, Phys. Rev. Lett. 24, 100 (1970).

[17]V. Narayanamurti and C. M. Varma, preceding Letter [Phys. Rev. Lett. 25, 1105 (1970)].

ANISOTROPY OF THE ELECTRON-IMPURITY SCATTERING IN SOME DILUTE GOLD ALLOYS

D. H. Lowndes,* K. Miller, and M. Springford
School of Mathematical and Physical Sciences, University of Sussex, Brighton, England
(Received 2 June 1970)

A general method is described for obtaining a map of the variation of the electronic lifetime over the entire Fermi surface using the Dingle temperatures measured in the de Haas–van Alphen effect. The method is illustrated by presenting maps of the variation of the electron-impurity scattering lifetime for Au:Ag and Au:Fe dilute alloys.

We wish to present some preliminary results in which, for the first time, a map showing the detailed variation of the electronic lifetime over an entire Fermi surface has been obtained. We describe a conceptually simple and general analytical method for decomposing the orbitally averaged lifetimes measured in de Haas–van Alphen (dHvA) experiments to yield local values of the lifetime. The method requires a detailed knowledge of the topography of the Fermi surface and of the distribution of electronic velocities over it. Such information has recently become available for the noble metals[1] and we illustrate the method using two dilute alloys of gold, Au:Fe and Au:Ag, in which the scattering of electrons is dominated by the solute.

The determination of local values of the electronic lifetime over the Fermi surface in a metal is a problem of considerable current interest as evidenced by the many aspects of this problem reviewed in the published proceedings of a recent international conference.[2] For such determination, one requires ideally a physical effect which arises from a local and well-defined group of carriers. The application of one such effect, magnetic-field–induced quantum states, to study most elegantly the anisotropy of electron-phonon scattering in copper was reported recently in this journal.[3] The tilted-field Gantmakher effect has similarly been used in potassium.[4] While in principle such effects may be employed to study the scattering of electrons by imperfections other than phonons, their application is confined to materials of high purity relative to those which may conveniently be studied by means of the dHvA effect. The work reported here, while at an early stage, is intended to illustrate that the dHvA method may be used to explore in some detail the anisotropy of electron-impurity scattering for dilute alloys of relatively high solute concentrations. Demands upon material characterization are thus less stringent for the dHvA method which has the further advantage that problems inherent in the preparation of