CURRENT-PHASE RELATIONSHIP IN SHORT SUPERCONDUCTING WEAK LINKS

Alexis Baratoff*
Department of Physics, Brown University, Providence, Rhode Island 02912

and

James A. Blackburn†
National Aeronautics and Space Adminstration Electronic Research Center, Cambridge, Massachusetts 02139

and

Brian B. Schwartz‡
Francis Bitter National Magnet Laboratory,§ Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 27 July 1970)

We propose a one-dimensional model of a superconducting link which is "weak" only insofar as it has a lower critical current than the superconductors on either side. According to appropriate solutions of the Ginzburg-Landau equations, the supercurrent J is a single-valued, odd function of the phase change φ across the link, which goes to zero at $\varphi=0$ and $\varphi=\pi$ and tends to the Josephson relation $J\propto\sin\varphi$ in the limit of a very weak link.

The importance of phase coherence in superconductivity was first emphasized by Josephson, who predicted that a resistanceless current could flow through a tunneling junction between two superconductors.[1] He derived a simple equation $j=j_m\sin\varphi$ relating the supercurrent density j to the difference φ between the phases of the superconducting order parameter Δ on either side of the junction. Actually the exact nature of the junction does not come into the derivation; only a low transmission coefficient for the barrier is required. This is particularly clear in subsequent work by De Gennes who represented the barrier by effective boundary conditions on Δ and derived the correct behavior of the maximum supercurrent j_m near the transition temperature T_c from the Ginzburg-Landau (GL) equations.[2,3]

Characteristic Josephson interference effects have been observed with normal metal "barriers"[4] and even superconducting "point" contacts,[5] as well as with tunneling junctions. The only obvious common feature of such "weak links" is that the contact has a much lower critical current than the bulk superconducters on either side.

By monitoring the total magnetic flux Φ enclosed in a bulk superconducting ring containing a weak link as a function of the applied flux Φ_e, Silver and Zimmerman[6] were able to show that a sufficiently weak contact exhibited a single-valued, reversible, smooth, periodic (period 2π), though not necessarily sinusoidal, current-phase relationship.

The fact that a phase change on the order of π can occur across a short weak link is not a trivial observation. In a naive picture one would treat the link as a narrow constriction of length a and assume that the supercurrent increases linearly with the superfluid momentum p_s, resulting in a phase difference $\varphi\approx p_s a/\hbar$ across the link. The maximum phase difference would occur for $p_s\approx\hbar/\xi$, where ξ is the temperature-dependent Ginzburg-Landau coherence length, giving $\varphi_{\max}\approx a/\xi$, which can be much less than π. The phase change measured in the experiment ranged between 0 and π. In this Letter we show how such a weak-link current-phase relationship can arise by presenting a soluble model which describes a superconducting contact which is "weak" only in its lower critical-current-carrying capacity.

In order to understand the Josephson-like behavior of weak links, we consider the following somewhat oversimplified, but soluble, one-dimensional model. The "weak link" occupies region B ($|x|<\frac{1}{2}a$) which is different from region A ($|x|>\frac{1}{2}a$) only insofar as it has a shorter mean free path l_B and hence a larger penetration depth $\lambda_B=\lambda_A/\gamma^{1/2}$ and a smaller coherence length $\xi_B=\xi A\gamma^{1/2}$, where $\gamma=\chi_B/\chi_A$, χ being Gorkov's universal function of the impurity parameter l/ξ_0 (mean free path/BCS coherence distance). The superconductor in region B has a smaller critical current density than regions A, but otherwise they have the same thermodynamic properties, e.g., transition temperature T_c, equilibrium energy gap, and bulk critical field H_c. A uniform current density is assumed to flow in the x direction, and screening effects are ignored. This provides a fair description of the behavior expected in the immediate vicinity of a real point contact where most of the phase change occurs. The weak-link parameter γ is perhaps best thought of as representing the decrease in the effective cross-sectional area available for supercurrent flow in the weak link compared with the bulk. γ is expected to be very small in practice.

Our analysis is based on appropriate one-dimensional solutions of the GL equations[3] which are strictly valid only near T_c but should, nevertheless, provide a qualitatively correct picture of the situation even if nonlocal effects are important. The magnitude $f(x)$ and phase $\varphi(x)$ of the reduced order parameter, $\Delta/\Delta_0=f(x)e^{i\varphi(x)}$, must satisfy the following nonlinear equations:

$$-\xi^2\left[\frac{d^2f}{dx^2}-f\left(\frac{d\varphi}{dx}\right)^2\right]-f+f^3=0, \quad (1)$$

$$j=-\frac{c}{4\pi\lambda^2}\frac{\Phi_0}{2\pi}f^2\frac{d\varphi}{dx}=-\frac{cH_c{}^2\xi^2}{\Phi_0}f^2\frac{d\varphi}{dx}. \quad (2)$$

Introducing scaled variables $X=x/\xi_A$ and $J=j\Phi_0/(cH_c{}^2\xi)$, we obtain

$$f_A''-J^2/f_A{}^3+f_A{}^3=0 \quad (|X|>d), \quad (3a)$$

$$\gamma f_B''-J^2/(\gamma f_B{}^3)+f_B-f_B{}^3=0 \quad (|X|<d), \quad (3b)$$

and

$$\varphi_c=2J\left(\int_0^d\frac{dX}{\gamma f_B{}^2}+\int_d^{d+X_c}\frac{dX}{f_A{}^2}\right), \quad (4)$$

where $d=a/(2\xi_A)$. The solutions of Eqs. (3a) and (3b) must be matched at the boundaries $X=\pm d$. The integral (4), defining the total phase change φ_c, is cut off at a distance X_c from the contact.[7] The most complete discussion of the relevant boundary conditions is due to Zaitsev.[8] In the case of our model they reduce to the continuity of Δ and of the normal component of $\chi\nabla\Delta$, i.e.,

$$f_A(d)=f_B(d)=f_d; \quad \varphi_A(d)=\varphi_B(d);$$
$$f_A'(d)=\gamma f_B'(d); \quad \varphi_A'(d)=\gamma\varphi_B'(d). \quad (5)$$

We have looked for symmetric solutions of (3a) and (3b), bounded between 0 and 1, and tending to a constant (f_∞) less than unity for $X>d$. f_∞ is related to the reduced current density through $J=f_\infty{}^2(1-f_\infty{}^2)^{1/2}$.

Both (3a) and (3b) can be integrated once to yield

$$(f_A')^2=2(f_\infty{}^2-f_A{}^2)^2[f_A{}^2-2(1-f_\infty{}^2)], \quad (6a)$$

$$\gamma(f_B')^2=2(f_B{}^2-f_0{}^2)[f_B{}^4-(2-f_0{}^2)f^2+2J^2/(\gamma f_0{}^2)], \quad (6b)$$

where $f_0=f_B(0)$. These equations can be used to replace the X integrations in (4) by more convenient integrations over f^2 and to express the boundary condition on f_A' and f_B' in terms of f_∞, f_0, and f_d. In addition, their solutions can be written in terms of known functions (see Table I) since the expressions on the right sides of (6a) and (6b) are cubic in f^2.[9] Such solutions were used by Mamaladze and Cheishvili in a study of superfluid flow through a porous partition[10] and by Langer and Ambegaokar in their work on fluctuations in one-dimensional superconductors.[7]

The solutions were matched by specifying d and γ, determining the value(s) f_0 satisfying the boundary condition relating f_0, f_d, and f_∞ for various values of f_∞ (or J). Typical results are shown in Fig. 1. Two acceptable solutions were found for $J<J_m(d,\gamma)$. This critical current is plotted as a function of γ for $d=1$ and $d=0.05$ in Fig. 2(b). We notice a dramatic enhancement of J_m above the intrinsic critical density $J_c\gamma^{1/2}$ of the impure superconductor in region B. Once

Table I. The solution to Eqs. (6) in regions A and B.

$$f_A{}^2=2(1-f_\infty{}^2)+(3f_\infty{}^2-2)\tanh^2\left[(X-d)(\tfrac{3}{2}f_\infty{}^2-1)^{1/2}+\tanh^{-1}\left(\frac{f_d{}^2-2(1-f_\infty{}^2)}{3f_\infty{}^2-2}\right)^{1/2}\right],$$

$$\Delta\equiv(1-f_0{}^2)^2-8J^2/(\gamma f_0{}^2), \quad q\pm\equiv\tfrac{3}{2}f_0{}^2-1\pm\tfrac{1}{2}\Delta^{1/2}, \quad p\equiv\{2[(f_0{}^4-f_0{}^2+j^2/(\gamma f_0{}^2)]\}^{1/2}.$$

Solution 1, $\Delta>0$, $q_+>q_-\geqslant 0$:

$$f_B{}^2=f^2+\left[\frac{\mathrm{sn}^2(u_1,k_1)}{\mathrm{cn}^2(u_1,k_1)}\right]q_-, \quad k_1{}^2=(q_+-q_-)/q_+, \quad u_1=X[q_+/(2\gamma)]^{1/2}.$$

Solution 2, $\Delta<0$:

$$f_B{}^2=f_0{}^2+p\left[\frac{1-\mathrm{cn}(u_2,k_2)}{1+\mathrm{cn}(u_2,k_2)}\right], \quad k_2{}^2=\tfrac{1}{2}\left[1+\frac{2-3f_0{}^2}{2p}\right], \quad u_2=X(2p/\gamma)^{1/2}.$$

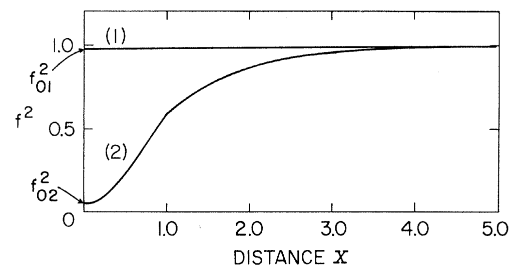

FIG. 1. The two solutions for $f^2(X)$ vs X for a weak link with $d=1.0$ and $\gamma=0.60$, and $f_\infty^2=0.9850$, $f_{01}^2=0.9761$, and $f_{02}^2=0.0451$.

f_0 and f_d are known, the phase difference can be obtained from (4). We actually calculated

$$\varphi = 2J\left[\int_0^d \frac{dX}{\gamma f_B^2(X)} + \int_d^\infty \frac{dX}{f_A^2(X)} - \int_0^\infty \frac{dX}{f_\infty^2}\right], \quad (7)$$

which is the phase difference between the solutions for a given J with and without the weak link, respectively.[11]

Typical results for J vs φ are shown in Fig. 2(a). The curves show the desired behavior and tend to the sinusoidal Josephson relationship for small γ.[12] As $\varphi \to \pi$, both f_0 and J approach zero, but J/f_0 tends to the finite limit $\gamma f_B'(0)$; the main contribution to the phase difference comes from the region B where $2J/\gamma f^2$ exhibits a δ-function-like peak of width $\gamma f_0^2/J$. Right at $\varphi = \pi$, the order order parameter vanishes in the middle of the weak link.

We wish to offer the following physical interpretation of our results. With increasing current flow the phase difference increases from 0. The supercurrent causes a marked depression of the order parameter in the region of the weak link. Since this depression propagates a few coherence lengths away from the contact, the weak link effectively acquires a length $\geqslant 2\xi_A$. The critical current of the weak link is enhanced as a result of this proximity effect. Thus the maximum supercurrent through the weak link can be much higher than in the previously discussed linear model, and corresponds to a phase change of the right order of magnitude no matter how short the link is. This explanation involves the nonlinear features of the theory in an essential way. The otherwise appealing derivation of Josephson-like behavior in point contacts proposed by Aslamazov and Larkin[13] approximates the spatial dependence of the order parameter in the region of the link by a harmonic function independent of the current and does not include such nonlinear effects.

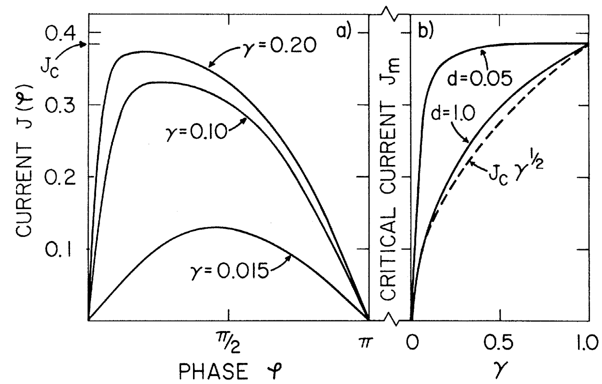

FIG. 2. (a) The current-phase relationship for a weak link with $d=0.05$ and $\gamma=0.20$, 0.10, and 0.015. Note that for $\gamma \ll 1$, the current-phase relationship becomes sinusoidal. (b) The critical current of the weak link as a function of its length and weakness parameter γ. Note that for $d \ll 1$, the critical current is not reduced from the bulk value until $\gamma \ll d$. The dashed line is the intrinsic critical current density of the weak-link material $J_c\gamma^{1/2}$.

Solutions for long links have also been found. When $d \gg 1$, or even $1 > d > \gamma^{1/2}$, the phase difference corresponding to the maximum current J_m is much larger than π, although φ still tends to 0 or π as $J \to 0$. The resulting $J(\varphi)$ curve is therefore re-entrant.

In order to avoid confusion we wish to note that Josephson-like behavior observed[14] in long links slightly below T_c has been explained in terms of averaging by thermal fluctuations between adjacent fluxoid quantum states,[6,15] a mechanism which has little in common with that required to explain observations at lower temperatures to which our arguments apply.

Extensive use was made of the digital computer facilities of the National Aeronautics and Space Administration Electronic Research Center. The authors wish to acknowledge helpful discussions with S. H. Autler, R. M. Meservey, and A. Schmid on various aspects of the problem.

---

*Work supported in part by the Advanced Research Projects Agency under Grant No. SD-86.

†National Academy of Sciences Research Associate. Present address: Goddard Institute for Space Studies, 2880 Broadway, New York, N. Y. 10025.

‡Also Department of Physics, Massachusetts Institute of Technology, Cambridge, Mass. 02139.

§Supported by the U. S. Air Force Office of Scientific Research.

[1]B. D. Josephson, Phys. Lett. 1, 251 (1962), and

Rev. Mod. Phys. 36, 216 (1964).

[2] P. G. De Gennes, Phys. Lett. 5, 22 (1963).

[3] P. G. De Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966), Chap. 7.

[4] J. D. Clarke, Proc. Roy. Soc., Ser. A 308, 447 (1969).

[5] J. E. Zimmerman and A. H. Silver, Phys. Rev. 141, 367 (1966).

[6] A. H. Silver and J. E. Zimmerman, Phys. Rev. 157, 317 (1967).

[7] J. S. Langer and V. Ambegaokar, Phys. Rev. 164, 498 (1967).

[8] R. O. Zaitsev. Zh. Eksp. Teor. Fiz. 50, 1055 (1966) [Sov. Phys. JETP 23, 702 (1966)].

[9] Referring to Table I we note that the solution in region A is such that $f_\infty^2 > \frac{2}{3}$ which means that the current density never reaches its critical value $J_c = 2\sqrt{3}/9$ in region A. It is possible to show analytically that the solution in region B is of type 1 for $1-f_0^2 \ll 1$ and of type 2 for $f_0 \ll 1$; it switches from one to the other at a value of $f_0^2 > \frac{2}{3}$ such that $J = \gamma^{1/2} f_0 (1-f_0^2)^{1/2}$. Other solutions were also found, but were rejected because they were not bounded (for $f_\infty^2 < \frac{2}{3}$), or could not be matched to the well-behaved solution in region A.

[10] Yu. G. Mamaladze and O. D. Cheishvili, Zh. Eksp. Teor. Fiz. 50, 169 (1966) [Sov. Phys. JETP 23, 112 (1966)].

[11] That $f_\infty < 1$ is an artifact of our one-dimensional model. Under actual experimental conditions, a weak link between bulk superconductors and the current flowing on the surface of the bulk superconductor, Δ should reach its equilibrium value ($f \to 1$ and $\varphi = \text{const}$) roughly within a coherence distance away from the junction. The cutoff X_c in expression (4) can be introduced to correct for this artifact. The actual $J(\varphi)$ curves should appear skewed to the right, as a result of a corresponding correction $2X_c(1-f_\infty^2)^{1/2}$ to the phase, but this has a negligible effect for sufficiently small γ, e.g., the slope $(dJ/d\varphi)_{\varphi=\pi}$ can be shown to be $\frac{1}{2}(X_c + d/\gamma)^{-1} \sim \gamma/2d$ for $d \gg \gamma$.

[12] It is not clear whether the portion of the $J(\varphi)$ curve beyond the maximum J_m corresponds to a physically realizable equilibrium solution. Theoretical considerations for a homogeneous superconductor carrying a uniform current show that the decreasing part of the corresponding $J(p_s)$ curve is unstable irrespective of load-line considerations. {See L. G. Aslamazov and A. I. Larkin, Pis'ma Zh. Eksp. Teor. Fiz. 9, 15 (1969) [JETP Lett. 9, 87 (1969)]. In the limit $\gamma = 1$, the corresponding part of our $J(\varphi)$ curve corresponds to the "saddle-point" solution considered by Langer and Ambegaokar, Ref. 7.} The relevant fluctuations may be stabilized in a short link, however [see A. Schmid, J. Low Temp. Phys. 1, 13 (1969)].

[13] Aslamazov and Larkin, Ref. 12.

[14] J. E. Lukens and J. M. Goodkind, Phys. Rev. Lett. 20, 1363 (1969).

[15] D. E. McCumber, Phys. Rev. 181, 716 (1969).

X-RAY DIFFRACTION DURING SHOCK-WAVE COMPRESSION*

Quintin Johnson, A. Mitchell, R. Norris Keeler, and L. Evans

Lawrence Radiation Laboratory, University of California, Livermore, California 94550

(Received 7 August 1970)

X-ray diffraction has been observed for the first time from material under shock compression. This was accomplished by directing a single pulse of x rays at LiF which was under compression from a shock wave, and observing the (200) diffraction line from the shock-compressed state. The experimental window for observing the effect was ~20 nsec; the pressure behind the shock front was ~130 kbar.

In recent work, Johnson, Keeler, and Lyle demonstrated that a Debye-Scherrer pattern can be produced in less than 100 nsec.[1] The x-ray device which is capable of this incorporates a pulsed x-ray generator built according to the principles of Blumlein[2] and Fitch and Howell.[3] The authors in Ref. 1 suggested that this device be applied to the study of materials undergoing shock compression; a report of first experiments towards that goal is published elsewhere.[4] Results of those experiments, while encouraging, were not conclusive because of the difficulty of reliably turning on the x-ray drive at the appropriate time. Using a Blumlein device which was modified to overcome this difficulty, however, we have clearly observed diffraction effects arising from the interaction of x rays and the shock-compressed state of LiF.

For this experiment, we made use of scintillation detectors described elsewhere.[5] These detectors, with a response time ~5 nsec, consisted of four channels, with the center point of each channel separated by 0.19 cm from its neighbor. The resolution of this detector system seriously limited accuracy but was sufficient to demonstrate without question that diffraction effects were observed from the shock-compressed state.

The experimental geometry is shown in Fig. 1. A high-explosive, plane-wave lens was boosted by a TNT pellet 1.27 cm thick. The resultant