and J. Ranft, Nuovo Cimento, Suppl. 6, 169 (1968).
[4]S. D. Drell, unpublished lecture.
[5]A. Liland and H. Pilkuhn, Phys. Lett. 29B, 663 (1969).
[6]J. R. Sanford and C. L. Wang, Brookhaven National Laboratory Reports No. BNL 11299 and No. BNL 11479, 1967 (unpublished).
[7]T. T. Chou and C. N. Yang, private communication. Also J. C. Vander Velde, to be published.
[8]D. Dekkers, J. A. Geibel, R. Mermod, G. Weber, T. R. Willitts, K. Winter, B. Jordan, M. Vivargent, N. M. King, and E. J. N. Wilson, Phys. Rev. 137, B962 (1965).
[9]R. A. Lundy, T. B. Novey, D. D. Yovanovitch, and V. L. Telegdi, Phys. Rev. Lett. 14, 504 (1966).
[10]J. G. Asbury, Y. Cho, M. Derrick, L. G. Ratner, T. P. Wangler, A. D. Krisch, and M. T. Lin, Phys. Rev. 178, 2086 (1969).
[11]G. Bellettini, G. Cocconi, A. N. Diddens, E. Lillethun, G. Matthiae, J. P. Scanlon, and A. M. Wetherell, Nucl. Phys. 79, 609 (1966).
[12]J. V. Allaby, F. Binon, A. N. Diddens, P. Duteil, A. Klovning, R. Meunier, J.-P. Peigneux, E. L. Sacharidis, K. Schluppmann, M. Spighel, J.-P. Stroot, A. M. Thorndike, and A. M. Wetherell, presented to the Fourteenth International Conference on High Energy Physics, Vienna, Austria, September 1968 (unpublished).
[13]E. W. Anderson, E. J. Bleser, G. B. Collins, T. Fujii, J. Menes, F. Turkot, R. A. Carrigan, Jr., R. M. Edelstein, N. C. Hien, T. J. McMahon, and I. Nadelhaft, Phys. Rev. Lett. 19, 198 (1967).
[14]V. S. Barashenkov and V. M. Maltsev, Fortschr. Phys. 15, 435 (1967).
[15]W. D. Walker, in *High Energy Collisions*, edited by C. N. Yang (Gordon and Breach, New York, 1970).
[16]W. F. Baker, R. L. Cool, W. E. Jenkins, T. F. Kycia, S. J. Lindenbaum, W. A. Love, D. Luers, J. A. Niederer, S. Ozaki, A. L. Read, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Lett. 7, 101 (1961).

REMARKS ABOUT THE HYPOTHESIS OF LIMITING FRAGMENTATION*

T. T. Chou and C. N. Yang
Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790
(Received 27 July 1970)

Remarks are made about the hypothesis of limiting fragmentation. In particular, the concept of favored and disfavored fragment distribution is introduced. Also a sum rule is proved leading to a useful quantity called energy-fragmentation fraction.

This paper contains a number of remarks about the recently proposed hypothesis[1] of limiting fragmentation.

Further experimental evidence. –Smith, Sprafka, and Anderson[2] have recently published a systematic study of the single-particle (π^- and π^+) distribution in pp collisions in a bubble chamber at five incoming energies from 13 to 28 BeV. This study remarkably confirms the hypothesis of limiting fragmentation. In fact, it seems that the single-particle distribution already approaches a limit in that region of incoming energies (see Fig. 1). [The main indication of this fact is that the coefficient $a_{\parallel}$ in Ref. 2 (for four to eight prongs) is experimentally proportional to $p_{\mathrm{inc}}^{-1/2}$.] We should emphasize that the Berkeley work[2] is the only published systematic comparative study of single-particle distributions for several incoming energies.

There have appeared[3] high-energy π^- spectrum measurements in p-Al collisions at 19 and 70 BeV/c. In the projectile system these represent slow "backward" π^- fragmented from the projectile proton (i.e., region A in Fig. 1). A comparison of the data at these two energies indicates that these backward π^--production differential cross sections fall with the incoming energy. We remark, however, that these differential cross sections are one or two orders of magnitude smaller than the main π^--production differential cross sections (which are "forward," i.e., in region B of Fig. 1, and form the bulk of the data of Ref. 2). That the approach to a limiting distribution is slow where the cross section is small is a general characteristic of all high-energy processes. [Cf. the approach to a limit of $d\sigma/dt$ for elastic processes, and Ref. 1, §3.]

It was emphasized many years ago[4] by Pal and Peters that the ratio μ^+/μ^- at sea level remains approximately 1.25 up to 100 BeV. They formulated from this fact a phenomenological model for high-energy collisions. Their model is in some essential respects consistent with the hypothesis of limiting fragmentation. (The main difference seems to be that in their model, the limiting distributions ρ_1, ρ_2, etc. exist, but their integrals are convergent rather

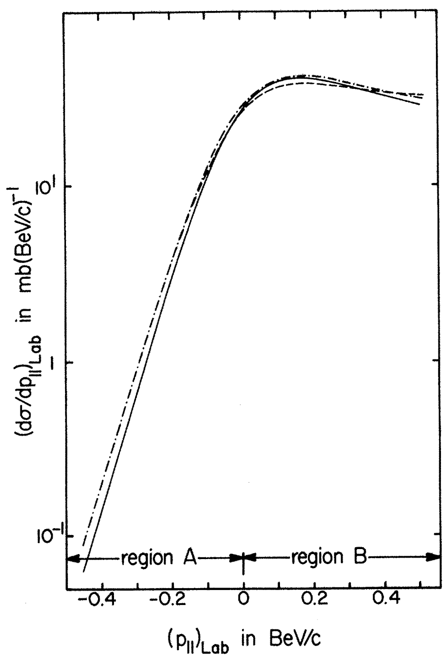

FIG. 1. $d\sigma/dp_{\parallel}$ distribution for π^- in pp collisions. These curves are superpositions of the four-, six-, and eight-prong fitted curves of Ref. 2, where we take all $p_\perp$ to be 0.2 BeV/c. This procedure is adopted since we lack the full original information for every event. It is estimated that the error is $\lesssim\pm5\%$. The dotted, dot-dashed, and solid curves are for 12.88-, 21.08-, and 28.44-BeV/c incoming momenta, respectively.

than divergent. See Ref. 1, §5. Thus in their model the increasing average multiplicity at high energies is attributed to a "fireball," while in the hypothesis of limiting fragmentation no fireball is needed to explain increasing multiplicities.) In particular their emphasis on the significance of the value of μ^+/μ^- at high energies is quite relevant for the hypothesis of limiting fragmentation: If the partial cross section for low multiplicities were to approach zero at high energies, the hypothesis of limiting fragmentation might become untenable. However, the ratio μ^+/μ^- would then approach unity at high muon energies, in contradiction to experimental facts.

A sum rule. —We shall adopt the notation

$$p \xrightarrow{\pi^-} p\pi^0\pi^0 \tag{1}$$

to mean the fragmentation of a proton into $p\pi^0\pi^0$ and nothing else under the impact of a π^- at infinite energies. In the process $p \xrightarrow{X}$ anything, one can prove that

$$[\sigma_{\mathrm{tot}}^{-1}\int\rho_1(\vec{p})M_t^{-1}(e-p_{\parallel})d^3p]_p+[\mathrm{same}]_n+[\mathrm{same}]_\Lambda+[\mathrm{same}]_{\pi^+}+\cdots=1, \tag{2}$$

where in the first term all quantities refer to a fragment proton [$\vec{p}$ is the momentum of a fragment proton, $\rho_1(\vec{p})$ is the fragment-proton limiting distribution, etc.], in the second term all quantities refer to the fragment neutron, etc. Generalization of (2) to the fragmentation of any hadron is obvious.

To prove (2) we remark that if in an event a proton fragments into m particles, the sum of $M_t^{-1}(e-p_{\parallel})$ over all m fragments is equal to 1 according to Eq. (I-10). We can then attribute to each fragment a fraction $x\equiv M_t^{-1}(e-p_{\parallel})$ of the event. Summing all such fractions for all fragment protons over a large number of events gives a quantity proportional to the first term of (2). Similar meaning can be given for other terms. (2) is then obvious. The following points are perhaps worth noticing:

(A) Equation (2) implies that each of the integrals on the left side is convergent. On the other hand, the integral[1] of ρ_1 is divergent:

$$\int\rho_1(\vec{p})[M_t^{-1}(e-p_{\parallel})]d^3p=\text{convergent}, \tag{3}$$

$$\int\rho_1(\vec{p})d^3p=\text{divergent}. \tag{4}$$

Since the divergence occurs in the region of large $p_{\parallel}$ and finite $p_\perp$, where $e-p_{\parallel}\propto(p_{\parallel})^{-1}$, (3) can be replaced by

$$\int\rho_1(\vec{p})(p_{\parallel})^{-1}d^3p=\text{convergent at large } p_{\parallel}. \tag{5}$$

Our speculation, from (4) and (5) and from the general trend of experimental data, is that

$$\rho_1(\vec{p})\propto(p_{\parallel})^{-1} \text{ for large } p_{\parallel}. \tag{6}$$

(6) suggests that the average fragmentation multiplicity $\langle n\rangle$ increases logarithmically with the incoming energy E_{inc}:

$$\langle n\rangle\propto\ln E_{\mathrm{inc}}. \tag{7}$$

One can limit oneself to fragmentations where the multiplicities remain less than a fixed number and define a $\rho_1(\vec{p})$ for such events, to be called $\rho_1'(\vec{p})$. Then the integral of $\rho_1'(\vec{p})$ is convergent, unlike (4). Instead of (6) one would have probably

$$\rho_1'(\vec{p})\propto(p_{\parallel})^{-2} \text{ for large } p_{\parallel}. \tag{8}$$

(B) One can also prove (2) by considering the

Table I. Rough estimate of fragmentation fraction for $p \xrightarrow{p} X\cdots$ for different fragments X. The estimate given includes the elastically scattered protons.

Fragment X	Fragmentation fraction
p	0.40
n	0.12
all π's	0.40
all K's	0.05

Table II. Experimentally observed favored and disfavored fragmentations.

Favored	Ref.	Disfavored	Ref.
$p \xrightarrow{p} p$	1, 5	$p \xrightarrow{p} \Lambda, K^\circ$	9
$\pi^- \xrightarrow{p} \pi^-$	6, 7	$p \xrightarrow{\pi^-} \Lambda, \pi^\pm, K^\circ$	6, 7
$\pi^+ \xrightarrow{p} \pi^+$	8	$\pi^- \xrightarrow{p} \pi^+$	6, 7
		$p \xrightarrow{p} \pi^\pm, K^\pm$	5

fragmenting proton as a projectile with energy $E_{\rm inc}=M_t\gamma$ which is very large. In the laboratory system the energy of the outgoing fragment is then $\gamma(e-p_\parallel)=E_{\rm inc}M_t^{-1}(e-p_\parallel)$. Thus $M_t^{-1}(e-p_\parallel)$ is the fraction of $E_{\rm inc}$ that becomes the energy of the fragment. The different terms in (2) are thus, respectively, the averages of such fractions for fragment p's, n's, Λ's, π^+'s, etc. We thus call the different terms in (2) the energy-fragmentation fractions, or simply fragmentation fractions, for p, n, Λ, π^+, etc., in $p \xrightarrow{X}$ anything. Notice that the first of these is the usually defined average "elasticity" of the proton. In Table I we list a very rough estimate of the fragmentation fraction in $p \xrightarrow{p}$ various particles.

Favored and disfavored fragment distributions. —Can one say anything about the fragment distributions? It seems that one can in those cases where one fragment is slow in the fragmenting particle's rest system and the others are all very fast. E.g., consider $p \to \pi^-\pi^+p$ where π^- is slow and the other fragments fast in the rest system of the fragmenting proton. Such a fragmentation is highly unlikely since it entails the large acceleration of two positive charges, a nucleon number, a total I of $\frac{3}{2}$, etc. This is indeed borne out by experimental information. [See Fig. 3(b) of Ref. 1. The τ_1 function drops rapidly to zero toward the left.]

Generalization of the argument leads to many qualitative features supported by various experiments. Stated concisely, the generalization asserts that for those fragmentations $\rho_n=\sigma_n+\tau_n$ where σ_n (a delta function) exists because the fragmentation satisfies certain selection rules (Ref. 1, §15), τ_n does not vanish near its kinematic boundary S_n [cf. (I-10)]; otherwise it vanishes. The former type of τ_n will be called favored, the latter disfavored; e.g., consider $\pi^-p \to \pi^\pm+\cdots$. For inelastic outgoing highest energy pions in the laboratory system, π^- is favored (since for π^- there exists a delta-function σ_1 due to elastic scattering), and π^+ is disfavored. This is indeed in excellent agreement with experiments,[6,7] and explains the dramatic difference in $\pi^\pm$ momenta distribution in π^-p interactions as exhibited, for example, in Figs. 16 and 17 of Ref. 6.

Other checks with experiments are tabulated in Table II.

If the concept of favored and disfavored fragment distribution is valid, one could make many interesting predictions. For example in the coherent process

$$\pi^- + \text{nucleus} \to \pi^-\pi^-\pi^+ + \text{nucleus}, \tag{9}$$

the most energetic outgoing π would rarely be a π^+. The average energy of the outgoing π^- would be higher than that of the outgoing π^+, etc. Notice that these are conclusions that are not natural in any resonance interpretation of (9).

In §9 of Ref. 1, an argument was given that for all cases τ_1 should vanish on the boundary. That argument was insufficient since it only proved that a fragmentation with a small Δ becomes increasingly unlikely. It does not prove that τ_1 should vanish, which is a stronger conclusion. We now no longer believe that τ_1 vanishes on the boundary for favored fragmentation.

Scaled variable x. —Many authors use[10] a scaled variable x equal to $p_\parallel^*/p_\parallel^*|_{\rm inc}$, where both p^*'s are measured in the c.m. system. For infinite incoming energy, Lorentz transformation trivally gives

$$x=M_t^{-1}(e-p_\parallel). \tag{10}$$

Thus for infinite energy, the transformation $(p_\parallel, \vec{p}_\perp) \to (x, \vec{p}_\perp)$ is a simple and finite one. Using x, the hypothesis of limiting fragmentation can be easily formulated in terms of distribution functions $\rho_1(x, \vec{p}_\perp)dxd^2p_\perp$, $\rho_2(x_1, \vec{p}_{1\perp}; x_2, \vec{p}_{2\perp})dx_1dx_2 \times d^2p_{1\perp}d^2p_{2\perp}$, etc. There are some advantages in so doing: Equation (I-10) becomes simply $\sum x=1$. The fragmentation fraction of, say, π^+ in the fragmentation of p becomes the average per collision of the sum of the x's of all π^+'s in the fragmentation process, etc.

But there is also a disadvantage in using x rather than the laboratory $p_{\parallel}$: The scaling of $p_{\parallel}^*$ involved in defining x renders obscure such geometrical concepts as the angle between two outgoing fragment momenta in the laboratory system. Thus correlations cannot be usefully discussed with the x variable.

Pionization. – Since there is confusion about the definition of pionization in the literature, a few remarks will be in order here. We adopt the following definition of pionization: Consider the fraction f of outgoing hadrons that have a c.m. energy less than a quantity W. For fixed W, let the incoming energy $E_{\rm inc}$ go to infinity. If f approaches a nonzero limit (for any W), we say there is pionization. If it approaches the limit zero, we say there is no pionization. [A more general form of the definition of pionization, which we shall not adopt, is to study the limit of f for $W = W_0(E_{\rm inc})^{a/2}$, where $0 \leqslant a < 1$.] The hypothesis of limiting fragmentation is consistent with either the presence or the absence of pionization although we favor the absence of pionization. In either case, all the particles included in $\rho_1(x, \vec{p}_\perp)dxd^2p_\perp$ [for example, $\rho_1 = x^{-1} \times \exp(-\beta p_\perp^2)$] are fragmentation products and not pionization products. [This fact was already emphasized in italics in §7 of Ref. 1. Notice that our definition of pionization is different from the definition of Cheng and Wu.[11] Their statement that the two definitions are the same is due to a misunderstanding.] If there is pionization, a very important question is whether there is always pionization or sometimes pionization, depending on how the fraction f fluctuates from event to event.

The variables ν and q^2. – The variables ν and q^2 used in inelastic e-p scattering are simply related to the laboratory variables $p_{\parallel}, \vec{p}_\perp$:

$$q^2 = (\sum \vec{p}_\perp)^2, \quad \nu = \sum p_{\parallel}, \tag{11}$$

where the sums extend over all fragments of the target and we have taken the limit $E_{\rm inc} \to \infty$. Similar variables in hadron-hadron collisions should also prove useful.

Definition of ρ_n. – For identical particles the definitions of ρ_n, σ_n, and τ_n given in Ref. 1 give, conveniently, a number of relations. As an example consider the case where all outgoing particles are π^0's. We have then

$$\rho_2(\vec{p}_1, \vec{p}_2) = \rho_2(\vec{p}_2, \vec{p}_1), \tag{12}$$

$$\int_{\rm all} \rho_n d^3p_1 \cdots d^3p_n = (n!)(\text{cross section for } n\pi^0), \tag{13}$$

$$\tau_1(\vec{p}_1) = \int \sigma_2(\vec{p}_1, \vec{p}_2) d^3p_2 + \frac{1}{2!}\int \sigma_3(\vec{p}_1, \vec{p}_2, \vec{p}_3) d^3p_2 d^3p_3 + \frac{1}{3!}\int \sigma_4 d^9p + \cdots, \tag{14}$$

etc.

---

*Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(30-1)3668B.

[1] J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, Phys. Rev. 188, 2159 (1969). Equations in this paper will be quoted as, for example, (I-10).

[2] D. B. Smith, R. J. Sprafka, and J. A. Anderson, Phys. Rev. Lett. 23, 1064 (1969).

[3] J. V. Allaby *et al.*, CERN Report No. 70-12 (unpublished); Yu. B. Bushnin *et al.*, Phys. Lett. 29B, 48 (1969); F. Binon *et al.*, Phys. Lett. 30B, 506 (1969).

[4] Y. Pal and B. Peters, Kgl. Danske Vidensk. Selsk., Mat.-Fys. Medd. 33, No. 15 (1964). We want to thank Professor Peters for a communication.

[5] Allaby *et al.*, Ref. 3.

[6] W. D. Walker, in *High Energy Collisions*, edited by C. N. Yang *et al.* (Gordon and Breach, New York, 1969).

[7] A. R. Erwin, in Proceedings of the Conference on Expectations for Particle Reactions at the New Accelerators, University of Wisconsin, Madison, Wisconsin, 1970 (unpublished).

[8] P. Franzini, private communication.

[9] R. Plano, private communication.

[10] R. P. Feynman, Phys. Rev. Lett. 23, 1415 (1969).

[11] H. Cheng and T. T. Wu, Phys. Rev. Lett. 23, 1311 (1969).